Algebras and their Associated Monomial Algebras*

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Abstract. Let $R = \oplus_{\Gamma\in\Gamma} R_\gamma$ be a Γ-graded K-algebra over a field K, where Γ is a totally ordered semigroup, and let I be an ideal of R. Considering the Γ-grading filtration FR of R and the Γ-filtration FA induced by FR for the quotient K-algebra $A = R/I$, we show that there is a Γ-graded K-algebra isomorphism $G(A) \cong \overline{A} = R/\langle \mathbf{HT}(I)\rangle$, where $G(A)$ is the associated Γ-graded K-algebra of A defined by FA, and $\langle \mathbf{HT}(I)\rangle$ is the Γ-graded ideal of R generated by the set of head terms of I. In the case that Γ is an ordered monoid with a well-ordering, this result enables us to lift many nice structural properties of $\overline{A}$ to A theoretically, and the natural connection with Gröbner basis theory leads to effective realization lifting information from the associated monomial algebras in both commutative and noncommutative cases.

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0. Introduction

Let K be a field, and let R be one of the following K-algebras:

- $K[x_1, ..., x_n]$, the commutative polynomial K-algebra in n variables, which has the standard K-basis $\mathcal{B} = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$;
- $K[a_1, ..., a_n]$, the K-algebra generated by $a_1, ..., a_n$ subject to the relations

$$a_j a_i = \lambda_{ji} a_i a_j, \quad \lambda_{ji} \in K^*,\ 1 \le i < j \le n,$$

  which has the standard K-basis $\mathcal{B} = \{a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$. (It is known that in computational algebra this algebra is studied as a typical solvable polynomial algebra (e.g., [K-RW], [Li1]),

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and in the study of noncommutative algebraic geometry it is called the coordinate ring of the n-dimensional quantum affine K-space.);

- $K\langle X_1, ..., X_n\rangle$, the noncommutative free K-algebra generated by $X = \{X_1, ..., X_n\}$, which has the standard K-basis $\mathcal{B} = \{1,\ X_{j_1}\cdots X_{j_s} \mid X_{j_i} \in X,\ s \geq 1\}$;
- KQ, the path algebra defined by a finite directed graph Q over K, which has the standard K-basis $\mathcal{B}$ consisting of all finite paths including vertices as paths of length 0.

Then, it is well-known that R holds a well-developed (commutative or noncommutative) Gröbner basis theory (cf. [Bu], [Mor], [K-RW], [Gr]). More precisely, let $\prec$ be a monomial ordeirng on $\mathcal{B}$. If I is an ideal of R, then, there is a (finite or infinite) Gröbner basis $\mathcal{G} \subset I$ in the sense that

$$\langle \mathbf{LM}(I)\rangle = \langle \mathbf{LM}(\mathcal{G})\rangle,$$

where $\langle \mathbf{LM}(I)\rangle$ is the ideal of R generated by the set of leading monomials $\mathbf{LM}(I)$ of I and $\langle \mathbf{LM}(\mathcal{G})\rangle$ is the ideal of R generated by the set of leading monomials $\mathbf{LM}(\mathcal{G})$ of $\mathcal{G}$ (see section 5 for the definition of a leading monomial). Put $A = R/I$ and $\overline{A} = R/\langle \mathbf{LM}(I)\rangle$. In the literature the K-algebra $\overline{A}$ is usually called the *associated monomial algebra* of the K-algebra A due to the fact that $\langle \mathbf{LM}(I)\rangle$ is a monomial ideal of R (e.g., see [An2], [G-IL], [G-I2], [GZ]). Historically, monomial algebras are studied and used widely in many mathematical areas such as algebraic geometry, representation theory of algebras, algebraic combinatorics, as such algebras may be understood more easily, and especially, may be manipulated on computer more effectively. To see the influence of the monomial algebra $\overline{A}$ on the algebra A, a motive example, which can be found in any computational work concerning Hilbert function, Hilbert series and Poincaré series of a (graded) algebra, is worthwhile to be recalled here. Let R be the free K-algebra $K\langle X_1, ..., X_n\rangle$, and let $\prec$ be a monomial ordering on $\mathcal{B}$. For an ideal I of R, put $A = R/I$, $\overline{A} = R/\langle \mathbf{LM}(I)\rangle$. Then the following statements hold.

(1) The image of the set $\mathcal{B} - \mathbf{LM}(I)$ in A, respectively in $\overline{A}$, forms a K-basis for A, respectively for $\overline{A}$.

(2) With respect to the natural $\mathbb{N}$-filtration on A and $\overline{A}$ (see section 1 for the definition), A and $\overline{A}$ have the same Hilbert function and hence have the same growth, or equivalently, A and $\overline{A}$ have the same Gelfand-Kirillov dimension.

(3) If I is an $\mathbb{N}$-graded ideal of R, then the $\mathbb{N}$-graded algebras A and $\overline{A}$ have the same Hilbert series.

(4) If $\mathcal{G}$ is a Gröbner basis with respect to $(\mathcal{B}, \prec)$, then all invariants in (1) – (3) are determined by $\mathbf{LM}(\mathcal{G}) \subset \langle \mathbf{LM}(I)\rangle$ and computable by means of some computer algebra system such as BERGMAN [CU].

Similar results hold for other commonly studied algebras that hold a Gröbner basis theory.

Based on the above review, we are naturally concerned about the following problem.

Question How to transfer as many as possible nice structural and computational properties of $\overline{A} = R/\langle \mathbf{LM}(I)\rangle$ to $A = R/I$.

In the case that R is the free K-algebra $K\langle X_1, ..., X_n\rangle$, if I is an ideal of R, $A = R/I$ is the quotient algebra of R defined by I, and $G^{\mathbb{N}}(A)$ is the associated $\mathbb{N}$-graded K-algebra of A with respect to its natural $\mathbb{N}$-filtration $F^{\mathbb{N}}A$ induced by the $\mathbb{N}$-grading filtration $F^{\mathbb{N}}R$ of R defined by its natura $\mathbb{N}$-gradation (see section 1 for the definition of a Γ-grading filtration), then it follows from ([Li1] Chapter III Proposition 3.1) that $G^{\mathbb{N}}(A) \cong R/\langle \mathbf{HT}(I)\rangle$ as $\mathbb{N}$-graded K-algebras, where $\langle \mathbf{HT}(I)\rangle$ is the $\mathbb{N}$-graded ideal of R generated by the set of head terms of I (see section 2 for the definition of a head term), and that if furthermore $\mathcal{G}$ is a Gröbner basis for I with respect to some graded monomial ordering on $\mathcal{B}$, then $\langle \mathbf{HT}(I)\rangle = \langle \mathbf{HT}(\mathcal{G})\rangle$. In [Li2], this result was extended to propose a general PBW property for quotient algebras of a $\mathbb{Z}$-graded algebra, and for quotient algebras of a path algebra (including free algebra), a solution to the general PBW problem is given by means of Gröbner bases. Enlightened by [Li2], in the present paper we strive for a solution to the problem posed above by virtue of the $\mathcal{B}$-filtration and Gröbner bases, but consideration is made in a more general setting. The contents of this paper are arranged as follows.

1. Γ-filtered Algebras and Modules
2. With Γ-grading Filtration: $G(R/I) \cong R/\langle \mathbf{HT}(I)\rangle$
3. Basic Lifting Properties
4. Lifting Homological Properties
5. With Gröbner Bases: $G^{\mathcal{B}}(R/I) \cong R/\langle \mathbf{LM}(\mathcal{G})\rangle$ & $G^{\mathbb{N}}(R/I) \cong R/\langle \mathbf{HT}(\mathcal{G})\rangle$
6. The First Application
7. Realization via Gröbner Bases and Ufnarovski Graphs

Convention throughout the paper

Let K be a field. All algebras considered are associative K-algebras with identity 1, and all modules, unless otherwise stated, are unitary left modules. Let R be a K-algebra and $S \subset R$. We write $\langle S\rangle$ for the (two-sided) ideal of R generated by the subset S, and write $\langle S]$ for the left ideal of R generated by S. Moreover, $K^* = K - \{0\}$.

Here we point out in advance that since the Γ-filtration is less studied in the literature, and due to the nontrivial difference between a general ordered semigroup Γ and $\mathbb{N}$, we introduce this notion and the associated Γ-graded structure in section 1 in a slightly detailed manner; besides, although all results of sections 3 – 4 are well-known in the case of $\Gamma = \mathbb{N}$, to convince the reader, we provide a detailed proof for each result concerning Γ-filtration, for, the author cannot say that all of them are just a trivial imitation of the $\mathbb{N}$-filtered case.

1. Γ-filtered Algebras and Modules

In this section, Γ denotes a *totally ordered* semigrouop, i.e., Γ is a semigroup on which there is a total ordering $\prec$ that is compatible with the binary operation of Γ in the sense that for γ_1, γ_2,

$\gamma \in \Gamma$,

$$\gamma_1 \prec \gamma_2 \text{ implies } \gamma\gamma_1 \prec \gamma\gamma_2 \text{ and } \gamma_1\gamma \prec \gamma_2\gamma.$$

Γ-filtered Algebra

A K-algebra A is said to be Γ-*filtered* if there is a family $FA = \{F_\gamma A\}_{\gamma\in\Gamma}$ consisting of K-subspaces $F_\gamma A$ of A, such that

(F1) $A = \cup_{\gamma\in\Gamma} F_\gamma A$,

(F2) $F_{\gamma_1}A \subseteq F_{\gamma_2}A$ if $\gamma_1 \preceq \gamma_2$,

(F3) $F_{\gamma_1}AF_{\gamma_2}A \subseteq F_{\gamma_1\gamma_2}A$, $\gamma_1, \gamma_2 \in \Gamma$.

If Γ has a smallest element γ_0, we also ask that $1 \in F_{\gamma_0}A$.

In the case that $\Gamma = \mathbb{Z}$ and A is a $\mathbb{Z}$-filtered K-algebra, if $F_nA = \{0\}$ for all $n < 0$, then A becomes an $\mathbb{N}$-filtered K-algebra, which is also called a *positively filtered* K-algebra.

In the definition given above, the family $FA = \{F_\gamma A\}_{\gamma\in\Gamma}$ is usually called a Γ-*filtration* of A.

Natural $\mathbb{N}$-filtration defined by lengths of monomials

Let $A = K[T]$ be a K-algebra generated by $T = \{a_i\}_{i\in J}$ over K. Then each element $a \in A$ can be written as a finite sum of the form

$$a = \sum \lambda_{i_1\cdots i_s} a_{i_1}^{\alpha_1} \cdots a_{i_s}^{\alpha_s}, \quad a_{i_j} \in T,\ \lambda_{i_1\cdots i_s} \in K,\ \alpha_i, s \in \mathbb{N},\ s \geq 1.$$

By abusing language, a nonzero element of the form $u = a_{i_1}^{\alpha_1} \cdots a_{i_s}^{\alpha_s}$ is called a *monomial* of A, and the *length* of u, denoted $l(u)$, is defined as $l(u) = \alpha_1 + \cdots + \alpha_s$. Let Ω be the set of all monomials in A, i.e., $\Omega = \{u = a_{i_1}^{\alpha_1} \cdots a_{i_s}^{\alpha_s} \mid a_{i_j} \in T,\ \alpha_i, s \in \mathbb{N},\ s \geq 1\}$. For each $p \in \mathbb{N}$, let F_pA denote the K-subspace of A spanned by all monomials of length less than or equal to p, that is

$$F_pA = K\text{-span}\left\{u \in \Omega \;\middle|\; l(u) \leq p\right\}.$$

It is easy to see that the family $FA = \{F_pA\}_{p\in\mathbb{N}}$ satisfies the foregoing conditions (F1)–(F3). This $\mathbb{N}$-filtration is called the *natural* $\mathbb{N}$-*filtration* of A defined by lengths of monomials.

Γ-grading filtration

Let R be a Γ-graded K-algebra, that is, $R = \oplus_{\gamma\in\Gamma} R_\gamma$, where for each $\gamma \in \Gamma$, R_γ is a K-subspace of R, and for any γ_1, $\Gamma_2 \in \Gamma$, $R_{\gamma_1}R_{\gamma_2} \subseteq R_{\gamma_1\gamma_2}$. Put

$$F_\gamma R = \bigoplus_{\gamma' \preceq \gamma} R_{\gamma'}, \quad \gamma \in \Gamma.$$

Then it may be checked directly that the family $FR = \{F_\gamma R\}_{\gamma\in\Gamma}$ satisfies the foregoing conditions (F1)–(F3). This Γ-filtration is called the Γ-*grading filtration* of R defined by the given Γ-gradation of R

Example (1) Let Let R be the free K-algebra $K\langle X_1, ..., X_n\rangle$, or the commutative polynomial K-algebra $K[x_1, ..., x_n]$, or the coordinate ring of the n-dimensional quantum affine K-space $K[a_1, ..., a_n]$, or the path algebra KQ defined by a finite directed graph Q over K, and let $\mathcal{B}$ be the standard K-basis of R. Then R is $\mathbb{N}$-graded by the $\mathbb{N}$-gradation $\{R_p\}_{p\in\mathbb{N}}$ with $R_p = K\text{-span}\{u \in \mathcal{B} \mid l(u) = p\}$. It is easy to see that the natural $\mathbb{N}$-filtration of R defined by lengths of monomials coincides with the $\mathbb{N}$-grading filtration defined by the $\mathbb{N}$-gradation $\{R_p\}_{p\in\mathbb{N}}$.

If furthermore $\prec$ is a monomial ordering on $\mathcal{B}$, then $\mathcal{B}$ becomes an ordered semigroup with the well-ordering $\prec$ (in the case that $R = KQ$, $\mathcal{B} \cup \{0\}$ is considered). It turns out that R is $\mathcal{B}$-graded, i.e., $R = \oplus_{u\in\mathcal{B}} R_u$ with $R_u = Ku$, and consequently, this $\mathcal{B}$-gradation defines the $\mathcal{B}$-grading filtration $FR = \{F_uR\}_{u\in\mathcal{B}}$ of R with $F_uR = \oplus_{u'\preceq u} R_{u'}$.

Both the $\mathcal{B}$-filtration and the $\mathbb{N}$-filtration of R will be used in later section 5.

The associated Γ-graded algebra

Let A be a Γ-filtered K-algebra with Γ-filtration $FA = \{F_\gamma A\}_{\gamma\in\Gamma}$. Put

$$F^*_\gamma A = \bigcup_{\gamma'\prec\gamma} F_{\gamma'}A, \quad \gamma \in \Gamma,$$
where $F^*_{\gamma_0}A = \{0\}$ if A has a smallest element γ_0.

The *associated* Γ*-graded* K*-algebra* of A, denoted $G(A)$, is defined as

$$G(A) = \bigoplus_{\gamma\in\Gamma} G(A)_\gamma \text{ with } G(A)_\gamma = F_\gamma A/F^*_\gamma A,$$

where the multiplication is defined by extending the maps

$$\begin{array}{ccc} G(A)_{\gamma_1} \times G(A)_{\gamma_2} & \longrightarrow & G(A)_{\gamma_1\gamma_2} \\ (\overline{a_{\gamma_1}}, \overline{a_{\gamma_2}}) & \mapsto & \overline{a_{\gamma_1}a_{\gamma_2}} \end{array}$$

to $G(A)\times G(A) \to G(A)$, in which $\overline{a_{\gamma_1}}$, $\overline{a_{\gamma_2}}$ are the images of $a_{\gamma_1} \in F_{\gamma_1}A$, $a_{\gamma_2} \in F_{\gamma_2}A$ in $G(A)_{\gamma_1} = F_{\gamma_1}A/F^*_{\gamma_1}A$ and $G(A)_{\gamma_2} = F_{\gamma_2}A/F^*_{\gamma_2}A$ respectively, and $\overline{a_{\gamma_1}a_{\gamma_2}}$ is the image of $a_{\gamma_1}a_{\gamma_2} \in F_{\gamma_1\gamma_2}A$ in $G(A)_{\gamma_1\gamma_2} = F_{\gamma_1\gamma_2}A/F^*_{\gamma_1\gamma_2}A$.

Γ-filtered module

Let A be a Γ-filtered K-algebra with Γ-filtration $FA = \{F_\gamma A\}_{\gamma\in\Gamma}$ and M an A-module. We say that M is a Γ-*filtered* A-*module* if there is a family $FM = \{F_\gamma M\}_{\gamma\in\Gamma}$ consisting of K-subspaces $F_\gamma M$ of M, such that

(FM1) $M = \cup_{\gamma\in\Gamma} F_\gamma M$,
(FM2) $F_{\gamma_1}M \subseteq F_{\gamma_2}M$ if $\gamma_1 \preceq \gamma_2$,
(FM3) $F_{\gamma_1}AF_{\gamma_2}M \subseteq F_{\gamma_1\gamma_2}M$, $\gamma_1, \gamma_2 \in \Gamma$.

In the definition given above, the family $FM = \{F_\gamma M\}_{\gamma\in\Gamma}$ is usually called a Γ-*filtration* of M.

Example (2) Given a Γ-filtered K-algebra A with Γ-filtration FA, if Γ has a smallest element γ_0 (for instance $\Gamma = \mathbb{N}$), then by the convention we made for FA, $1 \in F_{\gamma_0}A$. In this case, any A-module M has a Γ-filtration FM. To see this, let $\{\xi_i\}_{i\in J}$ be a generating set of M, i.e., $M = \sum_{i\in J} A\xi_i$. Put $V = \sum_{i\in J} F_{\gamma_0}A\xi_i$. Then it may be verified directly that the family $FM = \{F_\gamma M\}_{\gamma\in\Gamma}$ with $F_\gamma M = F_\gamma AV$ forms a Γ-filtration of M.

The associated Γ-graded module

Let A be a Γ-filtered K-algebra with Γ-filtration $FA = \{F_\gamma A\}_{\gamma\in\Gamma}$ and $G(A)$ the associated Γ-graded K-algebra of A. For a Γ-filtered A-module M with Γ-filtration $FM = \{F_\gamma M\}_{\gamma\in\Gamma}$, put

$$F^*_\gamma M = \textstyle\bigcup_{\gamma'\prec\gamma} F_{\gamma'}M, \quad \gamma \in \Gamma,$$

where $F^*_{\gamma_0}M = \{0\}$ if A has a smallest element γ_0.

The *associated* Γ*-graded module* of M, denoted $G(M)$, is the Γ-graded $G(A)$-module defined as

$$G(M) = \bigoplus_{\gamma\in\Gamma} G(M)_\gamma \text{ with } G(M)_\gamma = F_\gamma M/F^*_\gamma M,$$

where the module action is given by extending the maps

$$\begin{array}{ccc} G(A)_{\gamma_1} \times G(M)_{\gamma_2} & \longrightarrow & G(M)_{\gamma_1\gamma_2} \\ (\overline{a_{\gamma_1}}, \overline{m_{\gamma_2}}) & \mapsto & \overline{a_{\gamma_1}m_{\gamma_2}} \end{array}$$

to $G(A) \times G(M) \to G(M)$, in which $\overline{a_{\gamma_1}}$, $\overline{m_{\gamma_2}}$ are the images of $a_{\gamma_1} \in F_{\gamma_1}A$, $m_{\gamma_2} \in F_{\gamma_2}A$ in $G(A)_{\gamma_1} = F_{\gamma_1}A/F^*_{\gamma_1}A$ and $G(M)_{\gamma_2} = F_{\gamma_2}M/F^*_{\gamma_2}M$ respectively, and $\overline{a_{\gamma_1}m_{\gamma_2}}$ is the image of $a_{\gamma_1}m_{\gamma_2} \in F_{\gamma_1\gamma_2}M$ in $G(M)_{\gamma_1\gamma_2} = F_{\gamma_1\gamma_2}M/F^*_{\gamma_1\gamma_2}M$.

Γ-filtered submodule and induced Γ-filtration

Let A be a Γ-filtered K-algebra with Γ-filtration $FA = \{F_\gamma A\}_{\gamma\in\Gamma}$ and M a Γ-filtered A-module with Γ-filtration $FM = \{F_\gamma M\}_{\gamma\in\Gamma}$. If N is an A-submodule of M and N has a Γ-filtration $FN = \{F_\gamma N\}_{\gamma\in\Gamma}$ satisfying $F_\gamma N \subseteq F_\gamma M$ for all $\gamma \in \Gamma$, then we call N a Γ*-filtered* A*-submodule* of M. Indeed, any A-submodule N of M can be made into a Γ-filtered A-submodule by using the *induced* Γ*-filtration* FN consisting of

$$F_\gamma N = N \cap F_\gamma M, \quad \gamma \in \Gamma.$$

Furthermore, for an A-submodule N of M, the quotient A-module M/N has the *induced* Γ-filtration $F(M/N)$ consisting of

$$F_\gamma(M/N) = (F_\gamma M + N)/N, \quad \gamma \in \Gamma.$$

Γ-filtered homomorphism

Let A, B be Γ-filtered K-algebras with $FA = \{F_\gamma A\}_{\gamma\in\Gamma}$, $FB = \{F_\gamma B\}_{\gamma\in\Gamma}$, respectively. A K-algebra homomorphism φ: $A \to B$ is called a Γ*-filtered* K*-algebra homomorphism* if $\varphi(F_\gamma A) \subseteq F_\gamma B$ for all $\gamma \in \Gamma$.

Given a Γ-filtered K-algebra A and two Γ-filtered A-modules M, N with $FM = \{F_\gamma M\}_{\gamma\in\Gamma}$, $FN = \{F_\gamma N\}_{\gamma\in\Gamma}$, respectively. A Γ-*filtered A-homomorphism* from M to N is an A-module homomorphism ψ: $M \to N$ such that $\psi(F_\gamma M) \subseteq F_\gamma N$ for all $\gamma \in \Gamma$.

A Γ-filtered A-homomorphism ψ: $M \to N$ is said to be *strict* if it satisfies

$$\psi(F_\gamma M) = \psi(M) \cap F_\gamma N, \quad \gamma \in \Gamma.$$

Let M be a Γ-filtered A-module with Γ-filtration FM and N a submodule of M. Then, with respect to the filtration $FN = \{F_\gamma N = N \cap F_\gamma M\}_{\gamma\in\Gamma}$ of N and the filtration $F(M/N) = \{(F_\gamma M + N)/N\}_{\gamma\in\Gamma}$ of the quotient module M/N, induced by FM, the inclusion map $N \hookrightarrow M$ and the canonical map $M \to M/N$ are obviously strict Γ-filtered A-homomorphisms.

Verification of the following proposition is an easy exercise.

1.1. Proposition If ψ: $M \to N$ is a Γ-filtered A-homomorphism, then $V = \mathrm{Im}\psi$ is a Γ-filtered A-submodule of N with the Γ-filtration $FV = \{F_\gamma V = \psi(F_\gamma M)\}_{\gamma\in\Gamma}$, and $W = \mathrm{Ker}\psi$ is a Γ-filtered A-submodule of M with the induced filtration $FW = \{F_\gamma W = W \cap F_\gamma M\}_{\gamma\in\Gamma}$.

$\square$

The associated Γ-graded $G(A)$-homomorphism

If φ: $M \to N$ is a Γ-filtered A-homomorphism, then φ induces naturally a Γ-graded $G(A)$-homomorphism:

$$\begin{array}{rccc} G(\varphi): & G(M) = \displaystyle\bigoplus_{\gamma\in\Gamma} G(M)_\gamma & \longrightarrow & \displaystyle\bigoplus_{\gamma\in\Gamma} G(N)_\gamma = G(N) \\ & \sum \overline{m} & \mapsto & \sum \overline{\varphi(m)} \end{array}$$

where $\overline{m}$, respectively $\overline{\varphi(m)}$, is the image of $m \in F_\gamma M$ in $G(M)_\gamma$, respectively the image of $\varphi(m)$ in $G(N)_\gamma$.

Remark If we replace the field K by $\mathbb{Z}$, then the text of this section becomes that for Γ-filtered rings and Γ-filtered modules without any modification.

2. With Γ-grading Filtration: $G(R/I) \cong R/\langle \mathbf{HT}(I)\rangle$

Let $R = \oplus_{\gamma\in\Gamma} R_\gamma$ be a Γ-graded K-algebra, where Γ is a totally ordered semigroup with the total ordering $\prec$. In this section we establish, for an arbitrary ideal I and the quotient algebra $A = R/I$ defined by I, the Γ-graded K-algebra isomorphism $G(A) \cong R/\langle \mathbf{HT}(I)\rangle$ with respect to the Γ-filtration FA induced by the Γ-grading filtration FR as defined in section 1, where $\langle \mathbf{HT}(I)\rangle$ is the Γ-graded ideal of R generated by the set of head terms of I (see the definition of

a head term below). Besides, we conclude a similar result for an arbitrary left ideal L and the module $M = R/L$.

To start with, note that each element $f \in R$ can be written uniquely as a sum of finitely many homogeneous elements, say $f = \sum_{i=1}^{s} r_{\gamma_i}$, $r_{\gamma_i} \in R_{\gamma_i}$. Assuming $\gamma_1 \succ \gamma_2 \succ \cdots \succ \gamma_s$, we define the *head term* of f, denoted $\mathbf{HT}(f)$, to be r_{γ_1}, that is,

$$\mathbf{HT}(f) = r_{\gamma_1},$$

and say that f is of *degree* γ_1, denoted $d(f) = \gamma_1$. For a subset $S \subset R$, we define the set of head terms of S as

$$\mathbf{HT}(S) = \left\{ \mathbf{HT}(r) \;\middle|\; r \in S \right\}.$$

Let I be an ideal of R. As $\mathbf{HT}(I)$ consists of homogeneous elements, the ideal $\langle \mathbf{HT}(I) \rangle$ of R is Γ-graded, and hence, the quotient algebra $\overline{A} = R/\langle \mathbf{HT}(I) \rangle$ is a Γ-graded K-algebra with the Γ-gradation $\{\overline{A}_\gamma = (R_\gamma + \langle \mathbf{HT}(I) \rangle)/\langle \mathbf{HT}(I) \rangle\}_{\gamma\in\Gamma}$. Consider the Γ-grading filtration $FR = \{F_\gamma R\}_{\gamma\in\Gamma}$ of R in the sense of section 1, where

$$F_\gamma R = \bigoplus_{\gamma' \preceq \gamma} F_{\gamma'} R, \quad \gamma \in \Gamma.$$

Then the quotient algebra $A = R/I$ has the induced Γ-filtration $FA = \{F_\gamma A = (F_\gamma R + I)/I\}_{\gamma\in\Gamma}$. Taking the associated Γ-graded K-algebra $G(A)$ of A defined by FA into account, we have the following key result of this paper.

2.1. Theorem With notation as fixed above, we have a Γ-graded K-algebra isomorphism

$$G(A) \cong \overline{A} = R/\langle \mathbf{HT}(I) \rangle.$$

Proof First, recall that Γ is ordered by the total ordering $\prec$. By the definition of $G(A)$, for $\gamma \in \Gamma$, $G(A)_\gamma = F_\gamma A / F_\gamma^* A$ with $F_\gamma A = (F_\gamma R + I)/I$ and, as a K-subspace,

$$\begin{aligned} F_\gamma^* A = \bigcup_{\gamma' \prec \gamma} F_{\gamma'} A &= \bigcup_{\gamma' \prec \gamma} \frac{F_{\gamma'} R + I}{I} \\ &= \frac{\bigcup_{\gamma' \prec \gamma} F_{\gamma'} R + I}{I} = \frac{F_\gamma^* R + I}{I}. \end{aligned}$$

It turns out that there are canonical isomorphisms of K-subspaces

$$\frac{R_\gamma \oplus F_\gamma^* R}{(I \cap F_\gamma R) + F_\gamma^* R} = \frac{F_\gamma R}{(I \cap F_\gamma R) + F_\gamma^* R} \xrightarrow{\cong} G(A)_\gamma, \quad \gamma \in \Gamma,$$

and consequently, we can extend the natural epimorphisms of K-subspaces

$$\phi_\gamma : \; R_\gamma \longrightarrow \frac{R_\gamma \oplus F_\gamma^* R}{(I \cap F_\gamma R) + F_\gamma^* R}, \quad \gamma \in \Gamma,$$

to define a Γ-graded K-algebra epimorphism

$$\phi : \ R \longrightarrow G(A).$$

We claim that $\mathrm{Ker}\phi = \langle \mathbf{HT}(I)\rangle$. To see this, noticing $\langle \mathbf{HT}(I)\rangle$ is a Γ-graded ideal, it is sufficient to prove the equalities

$$\mathrm{Ker}\phi_\gamma = \langle \mathbf{HT}(I)\rangle \cap R_\gamma, \quad \gamma \in \Gamma.$$

Suppose $r_\gamma \in \mathrm{Ker}\phi_\gamma \subset R_\gamma$. Then $r_\gamma \in (I \cap F_\gamma R) + F^*_\gamma R$. If $r_\gamma \neq 0$, then, as a homogeneous element of degree γ, $r_\gamma = \mathbf{HT}(f)$ for some $f \in I \cap F_\gamma R$. This shows that $r_\gamma \in \langle \mathbf{HT}(I)\rangle \cap R_\gamma$. Hence $\mathrm{Ker}\phi_\gamma \subseteq \langle \mathbf{HT}(I)\rangle \cap R_\gamma$. Conversely, suppose $r_\gamma \in \langle \mathbf{HT}(I)\rangle \cap R_\gamma$. Then, as a homogeneous element of degree γ, $r_\gamma = \sum_{i=1}^{s} v_i \mathbf{HT}(f_i) w_i$, where v_i, w_i are homogeneous elements of R and $f_i \in I$. Write $f_i = \mathbf{HT}(f_i) + f_i'$ such that $d(f_i') \prec d(f_i)$, $i = 1, ..., s$. By the fact that Γ is an ordered semigroup with the total ordering $\prec$, we may see that the expression

$$r_\gamma = \sum_{i=1}^{s} v_i f_i w_i - \sum_{i=1}^{s} v_i f_i' w_i$$

satisfies $\sum_{i=1}^{s} v_i f_i w_i \in I \cap F_\gamma R$ and $\sum_{i=1}^{s} v_i f_i' w_i \in F^*_\gamma R$. This shows that $r_\gamma \in (I \cap F_\gamma R) + F^*_\gamma R$, i.e., $r_\gamma \in \mathrm{Ker}\phi_\gamma$. Hence, $\langle \mathbf{HT}(I)\rangle \cap R_\gamma \subseteq \mathrm{Ker}\phi_\gamma$. Summing up, we conclude the desired equalities $\mathrm{Ker}\phi_\gamma = \langle \mathbf{HT}(I)\rangle \cap R_\gamma$, $\gamma \in \Gamma$. $\square$

Remark Obviously, if I is a Γ-graded ideal of R, then $A = R/I = G(A)$ with respect to FA induced by the Γ-grading filtration FR of R.

We illustrate Theorem 2.1 by two classical examples.

Example (1) Let g be a K-Lie algebra with the K-basis $\{x_1, ..., x_n\}$ and the bracket product

$$[x_i,\ x_j] = \sum_{\ell=1}^{n} \lambda_{ij}^{\ell} x_\ell, \quad 1 \le i < j \le n,\ \lambda_{ij}^{\ell} \in K,$$

and let $U(\mathsf{g})$ be the universal enveloping algebra of g. Then $U(\mathsf{g}) = K\langle X\rangle/I$, where $K\langle X\rangle = K\langle X_1, ..., X_n\rangle$ is the free K-algebra generated by $X = \{X_1, ..., X_n\}$ over K, and the ideal I is generated by $\mathcal{G} = \{X_jX_i - X_iX_j - \sum_{\ell=1}^{n} \lambda_{ij}^{n} X_\ell \mid 1 \le i < j \le n\}$. If we consider the $\mathbb{N}$-grading filtration $FK\langle X\rangle$ of $K\langle X\rangle$ as defined in section 1, then $FK\langle X\rangle$ induces the natural $\mathbb{N}$-filtration $FU(\mathsf{g}) = \{(F_pK\langle X\rangle + I)/I\}_{p\in\mathbb{N}}$ of $U(\mathsf{g})$. Hence, by Theorem 2.1, $U(\mathsf{g})$ has the associated $\mathbb{N}$-graded algebra $G(U(\mathsf{g})) \cong K\langle X\rangle/\langle \mathbf{HT}(I)\rangle$. It is well-known that $\mathcal{G}$ is a Gröbner basis for the ideal I with respect to a graded monomial ordering on $K\langle X\rangle$ (see [Mor]). By [Li1], $\mathbf{HT}(\mathcal{G})$ is a Gröbner basis of $\langle \mathbf{HT}(I)\rangle$ (also see later section 5). Hence, $G(U(\mathsf{g})) \cong K[x_1, ..., x_n]$, the commutative polynomial K-algebra in n variables. So, the classical PBW theorem is recaptured.

(2) Let $A_n(K)$ be the n-th Weyl algebra, that is, $A_n(K) \cong K\langle X,\ Y\rangle/I$, where $K\langle X,\ Y\rangle = K\langle X_1, ..., X_n, Y_1, ..., Y_n\rangle$ is the free K-algebra generated by $X \cup Y = \{X_1, ..., X_n, Y_1, ..., Y_n\}$ over

K, and the ideal I is generated by the set $\mathcal{G}$ consisting of

$$\begin{array}{ll} X_iX_j - X_jX_i,\ Y_iY_j - Y_jY_i, & 1 \le i < j \le n, \\ Y_jX_i - X_iY_j - \delta_{ij} \text{ the Kronecker delta}, & 1 \le i, j \le n. \end{array}$$

If we consider the $\mathbb{N}$-grading filtration $FK\langle X,\ Y\rangle$ of $K\langle X,\ Y\rangle$ as defined in section 1, then $FK\langle X,\ Y\rangle$ induces the natural $\mathbb{N}$-filtration $F_pA_n(K) = \{(F_pK\langle X\rangle + I)/I\}_{p\in\mathbb{N}}$ of $A_n(K)$. Hence, by Theorem 2.1, $A_n(K)$ has the associated $\mathbb{N}$-graded algebra $G(A_n(K)) \cong K\langle X,\ Y\rangle/\langle \mathbf{HT}(I)\rangle$. It is equally well-known that $\mathcal{G}$ is a Gröbner basis for the ideal I with respect to a graded monomial ordering on $K\langle X,\ Y\rangle$ (see [Mor]). By [Li1], $\mathbf{HT}(\mathcal{G})$ is a Gröbner basis of $\langle \mathbf{HT}(I)\rangle$ (also see later section 5). Hence, the classical result $G(A_n(K)) \cong K[x_1, ..., x_n, y_1, ..., y_n]$ is also recaptured, where the latter is the commutative polynomial K-algebra in $2n$ variables.

In general, suppose the ideal I is generated by $F = \{f_i\}_{i\in J}$. Put $\mathbf{HT}(F) = \{\mathbf{HT}(f_i) \mid f_i \in F\}$. Then, naturally, we expect that the equality

$$(*) \qquad \langle \mathbf{HT}(I)\rangle = \langle \mathbf{HT}(F)\rangle$$

holds, and consequently we would have $G(A) \cong R/\langle \mathbf{HT}(F)\rangle$. In other words, the equality $(*)$ amounts to propose a general version of PBW theorem. To realize this property effectively in later section 5, let us examine how a generating set of I gives rise to a generating set for $\langle \mathbf{HT}(I)\rangle$, and vice versa.

2.2. Proposition Let $F = \{f_i\}_{i\in J}$ be a subset of the ideal I. The following two statements hold.

(i) Suppose that F is a generating set of I having the property that each nonzero $f \in I$ has a presentation

$$\begin{array}{rcl} f & = & \sum v_jf_jw_j,\ \text{in which } v_j,\ w_j \text{ are homogeneous elements of } R,\ f_j \in F, \\ & & \text{such that } v_jf_jw_j \ne 0 \text{ and } d(\mathbf{HT}(v_j\mathbf{HT}(f_j)w_j)) \preceq d(f), \\ & & \text{where } f_j \text{ may appear repeatedly.} \end{array}$$

Then $\langle \mathbf{HT}(I)\rangle = \langle \mathbf{HT}(F)\rangle$.

(ii) In the case that $\prec$ is a well-ordering on Γ, if $\langle \mathbf{HT}(I)\rangle = \langle \mathbf{HT}(F)\rangle$, then F is a generating set of I having the property mentioned in (i) above.

Proof (i) By definition, if $f \in R$, $f \ne 0$ and $\mathbf{HT}(f) \in R_\gamma$, then $d(f) = d(\mathbf{HT}(f)) = \gamma$. Since Γ is an ordered semigroup with the total ordering $\prec$, by the assumption on the presentation $f = \sum v_jf_jw_j$, the head term $\mathbf{HT}(f)$ of f must have the form $\mathbf{HT}(f) = \sum v_j\mathbf{HT}(f_j')w_j$, i.e., $\mathbf{HT}(f) \in \langle \mathbf{HT}(F)\rangle$. Hence $\langle \mathbf{HT}(I)\rangle = \langle \mathbf{HT}(F)\rangle$.

(ii) For $f \in I$, $f \ne 0$, suppose $\mathbf{HT}(f) \in R_\gamma$. By the assumption, the homogeneous element $\mathbf{HT}(f)$ can be written as

$$\mathbf{HT}(f) = \sum_{j=1}^{s} v_j\mathbf{HT}(f_j)w_j,$$

in which v_j, w_j are homogeneous of R and $f_j \in F$, satisfying $v_j\mathbf{HT}(f_j)w_j \neq 0$ and $d(v_j\mathbf{HT}(f_j)w_j) = d(\mathbf{HT}(f)) = d(f) = \gamma$, $j = 1, ..., s$, where $\mathbf{HT}(f_j)$ may appear repeatedly. Thus, writing each f_j as $f_j = \mathbf{HT}(f_j) + f'_j$ such that $d(f'_j) \prec d(f)$, we have

$$\mathbf{HT}(f) = \sum_{j=1}^{s} v_j f_j w_j - \sum_{j=1}^{s} v_j f'_j w_j,$$

in which each $v_j f_j w_j \neq 0$, $d(\sum_{j=1}^{s} v_j f_j w_j) = d(\mathbf{HT}(f)) = d(f) = \gamma$, and $d(\sum_{j=1}^{s} v_j f'_j w_j) \prec \gamma$. It turns out that the element

$$f' = f - \sum_{j=1}^{s} v_j f_j w_j \in I$$

has $d(f') \prec d(f) = \gamma$. For f', we may repeat the same procedure and get some

$$f'' = f' - \sum_{k=1}^{m} v_k f_k w_k \in I$$

with $d(f'') \prec \gamma'$, where v_k, w_k are homogeneous elements of R, $f_k \in F$, satisfying each $v_k f_k w_k \neq 0$ and $d(\sum_{k=1}^{m} v_k f_k w_k) = \gamma'$. Since $\prec$ is a well-ordering, after a finite number of repetitions, such reduction procedure of decreasing degrees must stop to give us an expression $f = \sum v_j f_j w_j$ with the desired property. $\Box$

We finish this section by remarking that the proof of Theorem 2.1 and Proposition 2.2 may be carried to deal with a left ideal L of R and the module $M = R/L$ directly so long as we replace I by L and consider only left-hand side action. We mention the result below but will not dig in detail on this topic in this paper.

2.3. Theorem Let L be an arbitrary left ideal of R and $M = R/L$ the quotient module of R determined by L. Considering the Γ-filtration $FL = \{F_\gamma L = L \cap F_\gamma R\}_{\gamma\in\Gamma}$ of L and the Γ-filtration $FM = \{F_\gamma M = (F_\gamma R + L)/L\}_{\gamma\in\Gamma}$ of M, induced by the Γ-grading filtration FR of R, let $G(M)$ be the associated Γ-graded $G(A)$-module. Then we have a Γ-graded R-isomorphism

$$G(M) \cong R/\langle\mathbf{HT}(L)],$$

where $\langle\mathbf{HT}(L)]$ denotes the Γ-graded left ideal of R generated by $\mathbf{HT}(L)$.

$\Box$

2.4. Proposition Let $F = \{f_i\}_{i\in J}$ be a subset of a left ideal L of R. The following two statements hold.

(i) Suppose that F is a generating set of L having the property that each nonzero $f \in L$ has a presentation

$$\begin{aligned} f \;=\; & \textstyle\sum v_j f_j, \text{ in which the } v_j \text{are homogeneous elements of } R,\ f_j \in F,\\ & \text{such that } v_j f_j \neq 0 \text{ and } d(\mathbf{HT}(v_j\mathbf{HT}(f_j))) \preceq d(f),\\ & \text{where } f_j \text{ may appear repeatedly.} \end{aligned}$$

Then $\langle \mathbf{HT}(I)] = \langle \mathbf{HT}(F)]$.
(ii) In the case that $\prec$ is a well-ordering on Γ, if $\langle \mathbf{HT}(I)] = \langle \mathbf{HT}(F)]$, then F is a generating set of L having the property mentioned in (i) above.

$\square$

3. Basic Lifting Properties

In this and the next section we explore the structural relation between Γ-filtered A-modules and Γ-graded $G(A)$-modules, where Γ is an ordered monoid with a well-ordering, that leads to many nice lifting properties. In view of section 2, these lifting properties provide us with a firm basis to study quotient algebras $A = R/I$ of a Γ-graded K-algebra R via the quotient algebra $R/\langle \mathbf{HT}(I)\rangle = G(A)$ with respect to the Γ-filtration FA induced by the Γ-grading filtration FR of R, especially when R is taken to be a free algebra, or a commutative polynomial algebra, or a path algebra, or some other commonly used graded algebra such as the coordinate rings of quantum affine K-spaces, Gröbner bases may be used to realize the lifting properties effectively (see later sections 5 – 7).

Let Γ be an ordered *monoid* with the *well-ordering* $\prec$. If γ_0 is the identity element of Γ, we assume that γ_0 is the *smallest* element in Γ. All notations used in previous sections are maintained.

Let A be a Γ-filtered K-algebra with Γ-filtration FA, and let $G(A)$ be the associated Γ-graded K-algebra of A. Then $1 \in F_{\gamma_0}A$ by the assumption on Γ made above and the convention fixed in the definition of FA (section 1).

Let M be a Γ-filtered A-module with Γ-filtration $FM = \{F_\gamma M\}_{\gamma\in\Gamma}$, and let $G(M)$ be the associated Γ-graded $G(A)$-module of M, in the sense of section 3. Since $\prec$ is a well-ordering on Γ, for each element $m \in M$, we define the *degree* of m, denoted $d(m)$, as

$$d(m) = \min\left\{\gamma \in \Gamma \;\middle|\; m \in F_\gamma M\right\}.$$

If $m \neq 0$ and $d(m) = \gamma$, then we write $\sigma(m)$ for the corresponding *nonzero* homogeneous element of degree γ in $G(M)_\gamma = F_\gamma M/F^*_\gamma M$.

As to the σ-element defined above, We first note an easily verified but useful property:

$$(\sigma) \qquad \forall\, a \in A,\; m \in M,\; \text{either } \sigma(a)\sigma(m) = 0 \text{ or } \sigma(a)\sigma(m) = \sigma(am).$$

We deal first with K-basis, divisors of zeros and primeness (semi-primeness). Recall that a ring S is a domain if S does not have divisors of zero; and S is called a prime (semi-prime) ring if $s_1Ss_2 \neq 0$ for any nonzero $s_1, s_2 \in S$ (if $sSs \neq 0$ for any nonzero $s \in S$).

3.1. Theorem Let A be an arbitrary Γ-filtered K-algebra with Γ-filtration FA, and let $G(A)$ be the associated Γ-graded K-algebra of A. The following statements hold, especially when $A = R/I$ and $G(A) = R/\langle \mathbf{HT}(I)\rangle$ as in Theorem 2.1.
(i) Suppose that $\{a_i\}_{i\in J}$ is a subset of A such that $\{\sigma(a_i)\}_{i\in J}$ forms a K-basis for $G(A)$, then $\{a_i\}_{i\in J}$ is a K-basis of A. Hence, if $G(A)$ is finite dimensional over K then so is A.
(ii) If $G(A)$ is a domain then so is A.
(iii) If $G(A)$ is a prime (semi-prime) ring then so is A.

Proof (i) We show first that $\{a_i\}_{i\in J}$ is K-linearly independent, namely, if $a = \sum_{j=1}^{s}\lambda_{i_j}a_{i_j} = 0$, where $\lambda_i \in K$ and $a_{i_j} \in \{a_i\}_{i\in J}$, then all coefficients $\lambda_{i_j} = 0$. To this end, assume that a_{i_1}, a_{i_2}, ..., a_{i_t}, $t \le s$, all have the same degree γ that is the highest degree among all terms in the linear expression of a. Then since $K \subseteq F_{\gamma_0}A$, taking the image of a in $G(A)_\gamma = F_\gamma A/F^*_\gamma A$ into account, we have

$$\lambda_{i_1}\sigma(a_{i_1}) + \lambda_{i_2}\sigma(a_{i_2}) + \cdots \lambda_{i_t}\sigma(a_{i_t}) = 0$$

in $G(A)_\gamma$. By the K-linear independence of $\{\sigma(a_i)\}_{i\in J}$, $\lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_t} = 0$. Similarly we assert that all other coefficients in the linear expression of a are equal to 0. This proves the K-linear independence of $\{a_i\}_{i\in J}$.

Next, we conclude that A, as a K-space, is spanned by $\{a_i\}_{i\in J}$. To see this, let $a \in F_\gamma A - F^*_\gamma A$, i.e., $d(a) = \gamma$. Then, by the assumption, $\sigma(a)$ can be written uniquely as a linear combination of $\sigma(a_i)$'s, say

$$\sigma(a) = \sum_{j=1}^{s}\lambda_{i_j}\sigma(a_{i_j}), \quad \lambda_{i_j} \in K,\ a_{i_j} \in \{a_i\}_{i\in J}.$$

As $\sigma(a)$ is a homogeneous element of degree γ in $G(A)$ and $K \subseteq G(A)_{\gamma_0} = F_{\gamma_0}A$, it follows that all homogeneous elements $\sigma(a_{i_j})$ in the linear expression of $\sigma(a)$ have the same degree γ. Thus, by the definition of a σ-element, we have

$$a' = a - \sum_{j=1}^{s}\lambda_{i_j}a_{i_j} \in F^*_\gamma A.$$

Suppose $a' \in F_{\gamma'}A - F^*_{\gamma'}A$, i.e., $d(a') = \gamma' \prec \gamma$. By a similar argument we may get $a'' = a' - \sum_{\ell=1}^{m}\lambda_{i_\ell}a_{i_\ell} \in F^*_{\gamma'}A$ with $\lambda_{i_\ell} \in K$ and $a_{i_\ell} \in \{a_i\}_{i\in J}$. If $d(a'') = \gamma''$, then $\gamma \succ \gamma' \succ \gamma''$. Since $\prec$ is a well-ordering, after repeating the procedure of reducing degrees for a finite number of steps we must have $a \in K\text{-span}\{a_i\}_{i\in J}$, as desired.
(ii) Let $a, b \in A$ be nonzero elements of degree γ_1, γ_2 respectively. Then $\sigma(a)$, $\sigma(b)$ are nonzero homogeneous elements of $G(A)$ and so $\sigma(a)\sigma(b) = \sigma(ab) \ne 0$. This means $ab \notin F^*_{\gamma_1\gamma_2}A$, and hence $ab \ne 0$.
(iii) If $a, b \in A$ are nonzero, then $\sigma(a)$, $\sigma(b)$ are nonzero homogeneous elements of $G(A)$ and so $\sigma(a)G(A)\sigma(b) \ne \{0\}$. It follows that there is a homogeneous element $\sigma(c) \in G(A)$, represented by $c \in A$, such that $\sigma(a)\sigma(c)\sigma(b) \ne 0$. But this means $acb \ne 0$. Hence $aAb \ne \{0\}$, i.e., A is prime. A similar argument holds for the semi-primeness. $\square$

Next, we focus on modules.

3.2. Proposition Let M be an A-module.
(i) Let $FM=\{F_\gamma M\}_{\gamma\in\Gamma}$ be a Γ-filtration of M. If $G(M)=\sum_{i\in J}G(A)\sigma(\xi_i)$ with $\xi_i\in M$ and $d(\xi_i)=\gamma_i\in\Gamma$, then $M=\sum_{i\in J}A\xi_i$ with

$$F_\gamma M=\sum_{i\in J}\left(\sum_{s\preceq\gamma,\ s_i\gamma_i=s}F_{s_i}A\right)\xi_i,\quad \gamma\in\Gamma.$$

In particular, if $G(M)$ is finitely generated then so is M.
(ii) If M is finitely generated, then M has a Γ-filtration $FM=\{F_\gamma M\}_{\gamma\in\Gamma}$ such that $G(M)$ is a finitely generated $G(A)$-module. Indeed, if $M=\sum_{i=1}^n A\xi_i$ and $\{\xi_1,...,\xi_n\}$ is a *minimal* set of generators for M, then the desired Γ-filtration FM consists of

$$F_\gamma M=\sum_{i=1}^n\left(\sum_{s\preceq\gamma,\ s_i\gamma_i=s}F_{s_i}A\right)\xi_i,\quad \gamma\in\Gamma,$$

where $\gamma_1,...,\gamma_n\in\Gamma$ are chosen arbitrarily.

Proof (i) Since $G(M)=\sum_{i\in J}G(A)\sigma(\xi_i)$ with $\xi_i\in M$ and $d(\xi_i)=\gamma_i\in\Gamma$, we have

$$G(M)_\gamma=\sum_{i\in J,\ \rho_i\gamma_i=\gamma}G(A)_{\rho_i}\sigma(\xi_i),\quad \gamma\in\Gamma.$$

Hence, for any $m\in F_\gamma M$, $m=\sum a_{\rho_i}\xi_i+m'$, where $a_{\rho_i}\in F_{\rho_i}A$, $\rho_i\gamma_i=\gamma$, and $m'\in F^*_\gamma M$. Assume $d(m')=\gamma'$. Then, similarly we have $m'=\sum a_{\mu_i}\xi_i+m''$, where $a_{\mu_i}\in F_{\mu_i}A$, $\mu_i\gamma_i=\gamma'$, and $m''\in F^*_{\gamma'}M$. Suppose $d(m'')=\gamma''$. Then, $\gamma\succ\gamma'\succ\gamma''$. As $\prec$ is a well-ordering, after repeating the procedure of reducing degrees for a finite number of steps, we should arrive at

$$m\in\sum_{i\in J}\left(\sum_{s\leq\gamma,\ s_i\gamma_i=s}F_{s_i}A\right)\xi_i.$$

Since m is taken arbitrarily, this shows that

$$F_u M=\sum_{i\in J}\left(\sum_{s\leq\gamma,\ s_i\gamma_i=s}F_{s_i}A\right)\xi_i,$$

and therefore $M=\sum_{i\in J}A\xi_i$.
(ii) Suppose $M=\sum_{i=1}^n A\xi_i$ and $\{\xi_1,...,\xi_n\}$ is a *minimal* set of generators for M. Choose $\gamma_1,...,\gamma_n\in\Gamma$ arbitrarily. Then, since each $m\in M$ has a presentation $m=\sum_{i=1}^n a_i\xi_i$ with $a_i\in F_{s_i}A-F^*_{s_i}A$ for some $s_i\in\Gamma$, it is easy to see that the K-subspaces

$$F_\gamma M=\sum_{i=1}^n\left(\sum_{s\leq\gamma,\ s_i\gamma_i=s}F_{s_i}A\right)\xi_i,\quad \gamma\in\Gamma,$$

form a Γ-filtration $FM = \{F_\gamma M\}_{\gamma\in\Gamma}$ for M. Furthermore, note that $1 \in F_{\gamma_0}A$, where γ_0 is the identity element of Γ. It follows from the construction of FM and the minimality of $\{\xi_1, ..., \xi_n\}$ (as a set of generators of M) that $\xi_i \in F_{\gamma_i}M - F^*_{\gamma_i}M$, i.e., $d(\xi_i) = \gamma_i$, $i = 1, ..., n$. Thus, by the foregoing property (σ) of the associated σ-elements, it is not difficult to verify that

$$G(M)_\gamma = F_\gamma M/F^*_\gamma M = \sum_{1\le i\le n,\ s_i\gamma_i=\gamma} G(A)_{s_i}\sigma(\xi_i), \quad \gamma \in \Gamma,$$

and hence $G(M) = \oplus_{\gamma\in\Gamma}G(M)_\gamma = \sum_{i=1}^n G(A)\sigma(\xi_i)$. $\square$

Recall that a sequence

$$\cdots \xrightarrow{\varphi_{i-2}} M_{i-1} \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \xrightarrow{\varphi_{i+1}} \cdots$$

of A-modules and A-homomorphisms is said to be *exact* if $\mathrm{Ker}\varphi_k = \mathrm{Im}\varphi_{k-1}$ holds for every k.

Γ-filtered homomorphisms and the associated Γ-graded homomorphisms considered below are in the sense of section 1.

3.3. Proposition Let

$$(*) \qquad L \xrightarrow{\varphi} M \xrightarrow{\psi} N$$

be a sequence of Γ-filtered A-modules and Γ-filtered A-homomorphisms satisfying $\psi \circ \varphi = 0$. Then the following properties are equivalent.
(i) The sequence $(*)$ is exact and φ and ψ are strict.
(ii) The associated sequence of Γ-graded $G(A)$-modules and Γ-graded $G(A)$-homomorphisms

$$G(*) \qquad G(L) \xrightarrow{G(\varphi)} G(M) \xrightarrow{G(\psi)} G(N)$$

is exact .

Proof For an element $x \in F_\gamma M$, we use $\overline{x}$ to denote the image of x in $F_\gamma M/F^*_\gamma M = G(M)_\gamma$. Similar notation is used for elements in L and N.
(i) $\Rightarrow$ (ii) By the definition of the associated Γ-graded $G(A)$-homomorphism of a Γ-filtered A-homomorphism, it is clear that $\mathrm{Im}G(\varphi) \subseteq \mathrm{Ker}G(\psi)$. To prove the converse inclusion, for $m \in F_\gamma M - F^*_\gamma M$, i.e., $d(m) = \gamma$, suppose $0 = G(\psi)(\overline{m}) = \overline{\psi(m)}$. If $\psi(m) = 0$, then $m \in \mathrm{Ker}\psi = \mathrm{Im}\varphi$ and there is some $\ell \in L$ such that

$$m = \varphi(\ell) \in \varphi(L) \cap F_\gamma M = \varphi(F_\gamma L).$$

Obviously we may assume $\ell \in F_\gamma L$, and thus, $\overline{m} = \overline{\varphi(\ell)} = G(\varphi)(\overline{\ell})$, i.e., $\overline{m} \in \mathrm{Im}G(\varphi)$. If $\psi(m) \neq 0$, then since $0 = G(\psi)(\overline{m}) = \overline{\psi(m)} \in G(N)_\gamma$, we have $\psi(m) \in F_{\gamma'}N - F^*_{\gamma'}N$ for some $\gamma' \prec \gamma$, i.e., $\psi(m) \in \psi(M) \cap F_{\gamma'}N = \psi(F_{\gamma'}M)$. This yields $\psi(m) = \psi(m')$ for some $m' \in F_{\gamma'}M$, and hence

$$m - m' \in \ \mathrm{Ker}\psi \cap F_\gamma M = \varphi(L) \cap F_\gamma M = \varphi(F_\gamma L).$$

Let $m - m' = \varphi(\ell')$ for some $\ell' \in F_\gamma L$. Then $\overline{m} = \overline{m - m'} = \overline{\varphi(\ell')} = G(\varphi)(\overline{\ell'})$. this shows that $\overline{m} \in \mathrm{Im}G(\varphi)$. As m is taken arbitrarily, so we have $\mathrm{Ker}G(\psi) \subseteq \mathrm{Im}G(\varphi)$. Therefore, we conclude $\mathrm{Ker}G(\psi) = \mathrm{Im}G(\varphi)$, that is, the sequence $G(*)$ is exact.
(ii) $\Rightarrow$ (i) Suppose that the graded sequence $G(*)$ is exact. Let us show that the sequence $(*)$ is exact first. If $\psi(m) = 0$ with $m \in F_\gamma M - F^*_\gamma M$, i.e., $d(m) = \gamma$, then $G(\psi)(\overline{m}) = 0$ with $\overline{m} = \sigma(m) \in G(M)_\gamma$. It follows that $\overline{m} = G(\varphi)(\overline{\ell'}) = \overline{\varphi(\ell')}$ for some $\ell' \in F_\gamma L - F^*_\gamma L$. Hence $m - \varphi(\ell') = m'$ for some $m' \in F_{\gamma'} M$ with $\gamma' \prec \gamma$. Thus, $\psi(m') = \psi(m - \varphi(\ell')) = 0$. Similarly, $m' - \varphi(\ell'') = m''$ with $\ell'' \in L$ and $m'' \in F_{\gamma''} M$ with $\gamma'' \prec \gamma'$. As the chain

$$\gamma \succ \gamma' \succ \gamma'' \succ \cdots$$

cannot be infinite, for $\prec$ is a well-ordering, after repeating the reduction procedure for a finite number of steps we arrive at $m = \varphi(\ell)$ for some $\ell \in L$. This shows that $\mathrm{Ker}\psi \subseteq \varphi(L)$. Therefore, $\mathrm{Ker}\psi = \varphi(L)$ and the exactness of the sequence $(*)$ follows.

As to the strictness of φ and ψ, we prove it only for ψ because a similar argument is valid for φ. Let $f \in F_\gamma N \cap \psi(M)$ and $f \notin F^*_\gamma N$. Then $f = \psi(m)$ for some $m \in F_w M$. Suppose $\gamma \preceq w$. If $w = \gamma$, then $f = \psi(m) \in \psi(F_\gamma M)$. If $\gamma \prec w$, then since $f \in F_\gamma N$, we have $G(\psi)(\overline{m}) = \overline{\psi(m)} = 0$ in $G(N)$. By the exactness, $\overline{m} = G(\varphi)(\overline{\ell}) = \overline{\varphi(\ell)}$ for some $\ell \in F_w L$. Put $m' = m - \varphi(\ell)$. Then $m' \in F_{w'} M$ with $w' \prec w$, and $\psi(m') = \psi(m - \varphi(\ell)) = \psi(m) = f$. If $\gamma \prec w'$, then similarly we may find $m'' \in F_{w''} M$ with $w'' \prec w'$, such that $\psi(m'') = f$. Note that the chain

$$w \succ w' \succ w'' \succ \cdots \succ \gamma$$

has finite length in Γ. So the reduction procedure above stops after a finite number of steps, and eventually we have $f = \psi(m_\gamma) \in \psi(F_\gamma M)$. This shows that $F_\gamma N \cap \psi(M) \subset \psi(F_\gamma M)$, that is, ψ is strict. $\square$

3.4. Corollary (i) Let φ: $M \to N$ be a Γ-filtered A-homomorphism. Then $G(\varphi)$ is injective, respectively surjective, if and only if φ is injective, respectively surjective, and φ is strict.
(ii) Let N, W be submodules of a Γ-filtered A-module M with Γ-filtration $FM = \{F_\gamma M\}_{\gamma\in\Gamma}$. Consider the Γ-filtration $FN = \{F_\gamma N = N \cap F_\gamma M\}_{\gamma\in\Gamma}$ of N and the Γ-filtration $FW = \{F_\gamma W = W \cap F_\gamma M\}_{\gamma\in\Gamma}$, induced FM respectively. If $N \subseteq W$, then $G(N) \subseteq G(W)$; and if $G(N) = G(W)$ then $N = W$.

$\square$

We summarize some immediate applications of previous results in the following theorem.

3.5. Theorem Let A be an arbitrary Γ-filtered K-algebra with Γ-filtration FA, and let $G(A)$ be the associated Γ-graded K-algebra of A. The following statements hold, especially when $A = R/I$ and $G(A) = R/\langle \mathbf{HT}(I)\rangle$ as in Theorem 2.1.

(i) Suppose that $G(A)$ is Γ-graded left Noetherian, that is, every Γ-graded left ideal of $G(A)$ is finitely generated, or equivalently, $G(A)$ satisfies the ascending chain condition for left ideals. Then every finitely generated A-module is left Noetherian, in particular, A is left Noetherian.
(ii) Suppose that $G(A)$ is Γ-graded left Artinian, that is, $G(A)$ satisfies the descending chain condition for left ideals. Then every finitely generated A-module is left Artinian, in particular, A is left Artinian.
(iii) If $G(A)$ is a Γ-graded simple K-algebra, that is, $G(A)$ does not have nontrivial ideal, then A is a simple K-algebra.
(iv) Let M be a Γ-filtered A-module with Γ-filtration FM. If the Krull dimension (in the sense of Gabriel and Rentschler) of $G(M)$ is well-defined, then the Krull dimension of M is defined and $\mathrm{K.dim} M \leq \mathrm{K.dim} G(M)$. In particular, this is true for $M = A$ and $G(M) = G(A)$.
(v) Let M be a Γ-filtered A-module with Γ-filtration FM. If $G(M)$ is a Γ-graded simple $G(A)$-module, that is, $G(M)$ does not have nontrivial Γ-graded submodule, then M is a simple A-module.
(vi) If $G(A)$ is semisimple (simple) Artinian, then A is semisimple (simple) Artinian.

Proof By the foregoing discussion, assertions (i) – (v) are clear. It remains to prove the semisimplicity of A in (vi). If A is Artinian, then it is well-known that the Jacobson radical $J(A)$ of A is nilpotent. As the semisimple ring $G(A)$ does not contain nilpotent ideal, if we use the Γ-filtration $FJ(A) = \{F_\gamma J(A) = J(A) \cap F_\gamma A\}_{\gamma \in \Gamma}$ of $J(A)$ induced by FA, then $G(J(A))$ is a Γ-graded ideal of $G(A)$ and hence $G(J(A)) = \{0\}$. By Corollary 5.4, $J(A) = \{0\}$ as desired. $\Box$

4. Lifting Homological Properties

In this section we keep the assumption that Γ is an ordered *monoid* with the *well-ordering* $\prec$, and the identity element γ_0 of Γ is the *smallest* element in Γ.

Let A be a Γ-filtered K-algebra with Γ-filtration $FA = \{F_\gamma A\}_{\gamma \in \Gamma}$, and let $G(A)$ be the associated Γ-graded K-algebra of A. With notation as before, the aim of this section is to lift several homological properties of $G(A)$ to A, in particular, when $A = R/I$ and $G(A) = R/\langle \mathbf{HT}(I) \rangle$ as in Theorem 2.1.

If $B = \oplus_{\gamma \in \Gamma} B_\gamma$ is a Γ-graded K-algebra and $M = \oplus_{\gamma \in \Gamma} M_\gamma$, $N = \oplus_{\gamma \in \Gamma} N_\gamma$ are Γ-graded B-modules, then, by a Γ-graded B-homomorphism from M to N we mean a B-homomorphism φ: $M \to N$ such that $\varphi(M_\gamma) \subseteq N_\gamma$, $\gamma \in \Gamma$.

We begin with some basics on graded free modules and graded projective modules.

Let $R = \oplus_{\gamma \in \Gamma} R_\gamma$ be a Γ-graded K-algebra. A Γ-*graded free* R-module is a free R-module $T = \oplus_{i \in J} Re_i$ on the basis $\{e_i\}_{i \in J}$, which is also Γ-graded such that each e_i is a homogeneous

element, that is, if $\deg(e_i)=\gamma_i$, $i\in J$, then $T=\oplus_{\gamma\in\Gamma}T_\gamma$ with

$$T_\gamma=\sum_{i\in J,\ w_i\gamma_i=\gamma}R_{w_i}e_i,\quad \gamma\in\Gamma.$$

By the definition, to construct a Γ-graded free R-module $T=\oplus_{i\in J}Re_i$ with the R-basis $\{e_i\}_{i\in J}$, it is sufficient to assign to each e_i a choosen degree.

Given any Γ-graded R-module $M=\oplus_{\gamma\in\Gamma}M_\gamma$, M has a generating set $\{m_i\}_{i\in J}$ consisting of homogeneous elements, i.e., $M=\sum_{i\in J}Rm_i$. Suppose $d(m_i)=\gamma_i$, $\gamma_i\in\Gamma$, $i\in J$. Then it is easy to see that

$$M_\gamma=\sum_{i\in J,\ w_i\gamma_i=\gamma}R_{w_i}m_i,\quad \gamma\in\Gamma.$$

Thus, considering the Γ-graded free R-module $T=\oplus_{i\in J}Re_i=\oplus_{\gamma\in\Gamma}T_\gamma$ with $d(e_i)=\gamma_i$, the map φ: $e_i\mapsto m_i$ defines a Γ-graded R-epimorphism φ: $T\rightarrow M$.

If T is a Γ-graded free R-module and P is a Γ-graded R-module, if there is another Γ-graded R-moduel Q such that $T=P\oplus Q$ and

$$T_\gamma=P_\gamma+Q_\gamma,\quad \gamma\in\Gamma,$$

then P is called a Γ-*graded projective* R-module.

Concerning graded projective modules, the following result is well-known (e.g., [NVO]).

4.1. Proposition For a Γ-graded R-module P, the following statements are equivalent.
(i) P is a Γ-graded projective R-module.
(ii) Given any exact sequence of Γ-graded R-modules and Γ-graded R-homomorphisms $M\xrightarrow{\psi}N\rightarrow 0$, if $P\xrightarrow{\alpha}N$ is a Γ-graded R-homomorphism, then there exists a unique Γ-graded R-homomorphism $P\xrightarrow{\varphi}M$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, "\varphi"'] \arrow[d, "\alpha"] & \\ M \arrow[r, "\psi"'] & N \arrow[r] & 0 \end{tikzcd}\qquad \psi\circ\varphi=\alpha$$

(iii) P is projective as an ungraded R-module.

$\square$

Returning to modules over the Γ-filtered K-algebra A with Γ-filtration FA, we first construct a Γ-*filtered free* A-*module* L with a Γ-filtration FL such that its associated Γ-graded module $G(L)$ is a Γ-graded free $G(A)$-module. To this end, let $L=\oplus_{i\in J}Ae_i$ be a free A-module on the basis $\{e_i\}_{i\in J}$. Then, as we did in section 3 (see the proof of Proposition 3.2(ii)), a Γ-filtration

FL for L can be constructed by using the A-basis $\{e_i\}_{i\in J}$ of L and arbitrarily chosen $\gamma_i \in \Gamma$, $i \in J$, that is,

$$F_\gamma L = \bigoplus_{i\in J}\left(\sum_{s\preceq\gamma,\ s_i\gamma_i=s} F_{s_i}A\right)e_i, \quad \gamma\in\Gamma.$$

4.2. Observation Note that now Γ is a monoid with the identity element γ_0 which is the smallest element in Γ. It is not difficult to see that in the construction of $FL = \{F_\gamma L\}_{\gamma\in\Gamma}$ above, for each $i \in J$, $e_i \in F_{\gamma_i}L - F^*_{\gamma_i}L$, that is, each e_i is of degree γ_i.

Convention In what follows, if we say that L is a Γ-filtered free A-module, then it is certainly the type constructed above.

4.3. Proposition The following statements hold.
(i) Let $L = \oplus_{i\in J}Ae_i$ be a Γ-filtered free A-module with Γ-filtration FL defined above, then the associated Γ-graded $G(A)$-module $G(L)$ of L is a Γ-graded free $G(A)$-module. More precisely, we have $G(L) = \oplus_{i\in J}G(A)\sigma(e_i) = \oplus_{\gamma\in\Gamma}G(L)_\gamma$ with

$$G(L)_\gamma = \sum_{i\in J,\ s_i\gamma_i=\gamma} G(A)_{s_i}\sigma(e_i), \quad \gamma\in\Gamma.$$

(ii) If $L' = \oplus_{i\in J}G(A)\eta_i$ is a Γ-graded free $G(A)$-module with the $G(A)$-basis $\{\eta_i\}_{i\in J}$ consisting of homogeneous elements, then there is some Γ-filtered free A-module L such that $L' \cong G(L)$ as Γ-graded $G(A)$-modules.
(iii) Let M be a Γ-filtered A-module with Γ-filtration $FM = \{F_\gamma M\}_{\gamma\in\Gamma}$. Then there is an exact sequence of Γ-filtered A-modules and strict Γ-filtered A-homomorphisms

$$0 \to N \xrightarrow{\iota} L \xrightarrow{\varphi} M \to 0$$

where L is a Γ-filtered free A-module with Γ-filtration FL, N is the kernel of the Γ-filtered A-epimorphism φ that has the Γ-filtration $FN = \{F_\gamma N = N\cap F_\gamma L\}_{\gamma\in\Gamma}$ induced by FL, and ι is the inclusion map.
(iv) If L is a Γ-filtered free A-module with Γ-filtration FL, N is a Γ-filtered A-module with Γ-filtration FN, and φ: $G(L) \to G(N)$ is a Γ-graded $G(A)$-epimorphism, then $\varphi = G(\psi)$ for some strict Γ-filtered A-epimorphism ψ: $L \to N$.

Proof (i) By the construction of FL, Observation 4.2 and the property (σ) of σ-elements formulated in section 3, the argument is straightforward.
(ii) Suppose $d(\eta_i) = \gamma_i$, $\gamma_i \in \Gamma$, $i \in J$. Then by (i) we see that the Γ-filtered free A-module $L = \oplus_{i\in J}Ae_i$ with $d(e_i) = \gamma_i$ satisfies $G(L) \cong L'$.
(iii) Let $\{\xi_i\}_{i\in J}$ be a generating set of M, that is, $M = \sum_{i\in J}A\xi_i$. Suppose $\xi_i \in F_{\gamma_i}M - F^*_{\gamma_i}M$, i.e., $d(\xi_i) = \gamma_i$, $i \in J$. Then the Γ-filtered free A-module $L = \oplus_{i\in J}Ae_i$ with $d(e_i) = \gamma_i$, $i \in J$, and the map φ: $e_i \mapsto \xi_i$ together make the desired exact sequence.

(iv) Let $L = \oplus_{i\in J}Ae_i$ be the Γ-filtered free A-module with $d(e_i) = \gamma_i$, $i \in J$. For each $i \in J$, choose $\xi_i \in F_{\gamma_i}N$ such that $\varphi(\sigma(e_i)) = \overline{\xi_i}$, where $\overline{\xi_i}$ is the homogeneous element in $G(N)_{\gamma_i}$ represented by ξ_i. Then ψ: $L \to N$ can be defined by putting

$$\psi\left(\sum a_ie_i\right) = \sum a_i\xi_i, \text{ where } \sum a_ie_i \in L.$$

Clearly, ψ is a Γ-filtered A-homomorphism. Since $G(\psi)$ and φ agree on generators, we have $G(\psi) = \varphi$. Hence, by Corollary 3.4, ψ is a strict Γ-filtered surjection. $\square$

4.4. Proposition Let P be a Γ-filtered A-module with Γ-filtration $FP = \{F_\gamma P\}_{\gamma\in\Gamma}$. The following statements hold.
(i) If $G(P)$ is a projective $G(A)$-module, then P is a projective A-module.
(ii) If $G(P)$ is a Γ-graded free $G(A)$-module, then P is a free A-module.

Proof (i) By Proposition 4.3(iii), there is an exact sequence of Γ-filtered A-modules and strict Γ-filtered A-homomorphisms

$$0 \longrightarrow N \xrightarrow{\iota} L \xrightarrow{\varphi} P \longrightarrow 0$$

where L is a Γ-filtered free A-module with Γ-filtration FL, N is the kernel of the Γ-filtered A-epimorphism φ that has the Γ-filtration $FN = \{F_\gamma N = N \cap F_\gamma L\}_{\gamma\in\Gamma}$ induced by FL, and ι is the inclusion map. It follows from Proposition 3.3 and Corollary 3.4 that the associated Γ-graded sequence

$$0 \longrightarrow G(N) \xrightarrow{G(\iota)} G(L) \xrightarrow{G(\psi)} G(P) \longrightarrow 0$$

is exact. Since P is a projective $G(A)$-module, by Proposition 4.1, this sequence splits through Γ-graded $G(A)$-homomorphisms. Consequently, $G(L) = G(P) \oplus G(N)$ with $G(L)_\gamma = G(P)_\gamma \oplus G(N)_\gamma$, $\gamma \in \Gamma$, and the projection of $G(L)$ onto $G(N)$ gives a Γ-graded $G(A)$-epimorphism ψ: $G(L) \to G(N)$ such that $\psi \circ G(\iota) = 1_{G(N)}$. Further, by Proposition 4.3(iv), $\psi = G(\beta)$ for some strict Γ-filtered A-epimorphism β: $L \to N$. Note that $G(\beta) \circ G(\iota) = G(\beta \circ \iota) = 1_{G(N)}$. It follows from Corollary 3.4 that $\beta \circ \iota$ is an automorphism of N. Hence, $L \cong K \oplus P$. This shows that P is projective.
(ii) Suppose $G(P) = \oplus_{i\in J}G(A)\sigma(\xi_i)$ with the free $G(A)$-basis $\{\sigma(\xi_i)\}_{i\in J}$, where each $\xi_i \in F_{\gamma_i}P - F^*_{\gamma_i}P$, i.e., $d(\xi_i) = \gamma_i \in \Gamma$, $i \in J$. Then, by Proposition 3.2 (or its proof), $P = \sum_{i\in J} A\xi_i$ with

$$F_\gamma P = \sum_{i\in J}\left(\sum_{s\preceq\gamma,\ s_i\gamma_i=s} F_{s_i}A\right)\xi_i, \quad \gamma \in \Gamma.$$

We claim that $\{\xi_i\}_{i\in J}$ is a free basis for P over A. To see this, construct the Γ-filtered free A-module $L = \oplus_{i\in J}Ae_i$ with Γ-filtration

$$F_\gamma L = \bigoplus_{i\in J}\left(\sum_{s\preceq\gamma,\ s_i\gamma_i=s} F_{s_i}A\right)e_i, \quad \gamma \in \Gamma,$$

as before, such that each e_i has the degree $\gamma_i = d(\xi_i)$. Then we have an exact sequence of Γ-filtered A-modules and strict Γ-filtered A-homomorphisms

$$0 \longrightarrow N \longrightarrow L \xrightarrow{\varphi} P \longrightarrow 0$$

where N has the Γ-filtration $FN = \{F_\gamma N = N \cap F_\gamma L\}_{\gamma\in\Gamma}$ induced by FL. It follows from Proposition 3.3 that this sequence yields an exact sequence of Γ-graded $G(A)$-modules and Γ-graded $G(A)$-homomorphisms

$$0 \longrightarrow G(N) \longrightarrow G(L) \xrightarrow{G(\varphi)} G(P) \longrightarrow 0$$

However, $G(\varphi)$ is an isomorphism. Hence $G(N) = \{0\}$ and consequently $N = \{0\}$ by Corollary 3.4. This proves that φ is an isomorphism, or in other words, P is free. $\square$

4.5. Proposition Let M be a Γ-filtered A-module with Γ-filtration $FM = \{F_\gamma\}_{\gamma\in\Gamma}$, and let

$$0 \to N' \to L'_n \to \cdots \to L'_0 \to G(M) \to 0 \tag{1}$$

be an exact sequence of Γ-graded $G(A)$-modules and Γ-graded $G(A)$-homomorphisms, where the L'_i are Γ-graded free $G(A)$-modules. The following statements hold.
(i) There exists an exact sequence of Γ-filtered A-modules and strict Γ-filtered A-homomorphisms

$$0 \to N \to L_n \to \cdots \to L_0 \to M \to 0 \tag{2}$$

in which the L_i are Γ-filtered free A-modules, such that we have the isomorphism of chain complexes

$$\begin{array}{ccccccccccc} 0 \to & N' & \to & L'_n & \to & \cdots & \to & L'_0 & \to & G(M) & \to 0 \\ & \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & & \downarrow= & \\ 0 \to & G(N) & \to & G(L_n) & \to & \cdots & \to & G(L_0) & \to & G(M) & \to 0 \end{array}$$

(ii) If N' is a projective $G(A)$-module, then N is a projective A-module; If N' is a Γ-graded free $G(A)$-module, then N is a free A-module.
(iii) If all modules in the sequence (1) are finitely generated over $G(A)$, then all modules in the sequence (2) are finitely generated over A.

Proof (i) By Proposition 4.3, the homomorphism $L'_0 \to G(M)$ in sequence (1) has the form $G(\beta)$ for some strict Γ-filtered surjection β: $L_0 \to M$, where L_0 is a Γ-filtered free A-module such that $L'_0 \cong G(L_0)$ as Γ-graded $G(A)$-modules. Let $N_0 = \mathrm{Ker}\beta$ and consider the Γ-filtration $FN_0 = \{F_\gamma N_0 = N_0 \cap F_\gamma L_0\}_{\gamma\in\Gamma}$ induced by FL_0. Then we have the diagram of Γ-graded $G(A)$-modules and Γ-graded $G(A)$-homomorphisms

$$\begin{array}{ccccccccc} \cdots \to & L'_2 & \to & L'_1 & \to & L'_0 & \to & G(M) & \to 0 \\ & & & & & \downarrow\cong & & \downarrow= & \\ & 0 & \to & G(N_0) & \to & G(L_0) & \to & G(M) & \to 0 \end{array}$$

which has two exact rows. Note that the directed square involved in the above diagram commutes. It turns out that the homomorphism $L'_1 \to L'_0$ factors through $G(N_0)$, that is, we obtain the diagram

$$\begin{array}{ccccccccc}
\cdots & \to & L'_2 & \to & L'_1 & \to & L'_0 & \to & G(M) & \to 0 \\
 & & & & \downarrow & & \cong\downarrow & & =\downarrow & \\
 & & 0 & \to & G(N_0) & \to & G(L_0) & \to & G(M) & \to 0 \\
 & & & & \downarrow & & & & & \\
 & & & & 0 & & & & &
\end{array}$$

in which both rows are exact and both directed squares commute. Starting with $L'_1 \to G(N_0) \to 0$, the foregoing constructive procedure can be repeated step by step to yield the desired sequence (2).

(ii) and (iii) follow immediately from Proposition 4.3 and Proposition 4.4, respectively. $\square$

To deal with flat modules over a Γ-filtered K-algebra A, we need to define a Γ-filtration, respectively a Γ-gradation, for a tensor product of two Γ-filtered A-modules, respectively for a tensor product of two Γ-graded $G(A)$-modules.

Let M be a Γ-filtered left A-module with Γ-filtration FM, and let N be a Γ-filtered right A-module with Γ filtration FN. Viewing $N \otimes_A M$ as a $\mathbb{Z}$-module, we define the Γ-filtration $F(N \otimes_A M)$ of $N \otimes_A M$ as

$$F_\gamma(N \otimes_A M) = \mathbb{Z}\text{-span}\left\{x \otimes y \;\middle|\; x \in F_vN,\ y \in F_wM \text{ and } vw \preceq \gamma\right\}, \quad \gamma \in \Gamma.$$

The associated Γ-graded $\mathbb{Z}$-module of $N \otimes_A M$ with respect to $F(N \otimes_A M)$ is then defined as $G(N \otimes_A M) = \oplus_{\gamma\in\Gamma} G(N \otimes_A M)_\gamma$ with

$$G(N \otimes_A M)_\gamma = F_\gamma(N \otimes_A M)/F^*_\gamma(N \otimes_A M), \quad \gamma \in \Gamma,$$

where $F^*_\gamma(N \otimes_A M) = \cup_{\gamma' \prec \gamma} F_{\gamma'}(N \otimes M)$.

Let P be a Γ-graded left $G(A)$-module, and let Q be a Γ-graded right $G(A)$-module. Viewing $Q \otimes_{G(A)} P$ as a $\mathbb{Z}$-module, we define the Γ-gradation of $Q \otimes_{G(A)} P$ as

$$(Q \otimes_{G(A)} P)_\gamma = \mathbb{Z}\text{-span}\left\{z \otimes t \;\middle|\; z \in Q_v,\ t \in P_w \text{ and } vw = \gamma\right\}, \quad \gamma \in \Gamma.$$

4.6. Lemma Let M be a Γ-filtered left A-module with Γ-filtration FM, and let N be a Γ-filtered right A-module with Γ filtration FN. With the definition made above, the following statements hold.

(i) For $\overline{x}_v \in G(N)_v$ represented by $x \in F_vN$, and $\overline{y}_w \in G(M)_w$ represented by $y \in F_wM$, the mapping defined by

$$\begin{array}{rccc}
\varphi(M,N): & G(N) \otimes_{G(A)} G(M) & \longrightarrow & G(N \otimes_A M) \\
 & \overline{x}_v \otimes \overline{y}_w & \mapsto & (\overline{x \otimes y})_{vw}
\end{array}$$

is an epimorphism of Γ-graded $\mathbb{Z}$-modules.

(ii) The canonical A-isomorphisms

$$A\otimes_A M \xrightarrow{\cong} M \text{ and } N\otimes_A A \xrightarrow{\cong} N$$

are strict Γ-filtered A-isomorphisms.

(iii) The strict Γ-filtered A-isomorphisms in (ii) induce Γ-graded $G(A)$-isomorphisms

$$G(A\otimes_A M) \xrightarrow{\cong} G(M) \text{ and } G(N\otimes_A A) \xrightarrow{\cong} G(N).$$

(iv) The canonical $G(A)$-isomorphisms

$$G(A)\ \otimes_{G(A)} G(M) \xrightarrow{\cong} G(M) \text{ and } G(N)\otimes_{G(A)} G(A) \xrightarrow{\cong} G(N)$$

are Γ-graded $G(A)$-isomorphisms.

Proof Verification is straightforward. $\square$

4.7. Proposition Let M be a Γ-filtered left A-module with Γ-filtration FM. If $G(M)$ is a flat Γ-graded $G(A)$-module, then M is a flat A-module.

Proof Let J be a right ideal of A and $FJ=\{F_\gamma J=J\cap F_\gamma A\}_{\gamma\in\Gamma}$ the Γ-filtration of J induced by FA. Consider the inclusion map ι: $J\hookrightarrow A$. Then the strict exactness of the Γ-filtered sequence

$$0\longrightarrow J\xrightarrow{\iota} A$$

yields the exact Γ-graded sequence

$$0\longrightarrow G(J)\xrightarrow{G(\iota)} G(A)$$

Furthermore, it follows from the flatness of $G(M)$ and Lemma 4.6 that we have the following commutative diagram of Γ-graded $\mathbb{Z}$-modules and Γ-graded $\mathbb{Z}$-homomorphisms:

$$\begin{tikzcd}
0 \arrow[r] & G(J)\otimes_{G(A)} G(M) \arrow[r, "G(\iota)\otimes 1_{G(M)}"] \arrow[d, "\varphi(M,J)"'] & G(A)\otimes_{G(A)} G(M) \arrow[d, "\varphi(M,A)"'] \arrow[dr, "\cong"] & \\
 & & & G(M) \arrow[dl, "\cong"] \\
 & G(J\otimes_A M) \arrow[r, "G(\iota\otimes 1_M)"'] & G(A\otimes_A M) &
\end{tikzcd}$$

As $\varphi(M,A)$ is an isomorphism and $G(\ell)\otimes 1_{G(M)}$ is a monomorphism, it turns out that $\varphi(M,J)$ is an isomorphism. So $G(\iota\otimes 1_M)$ must be a monomorphism. By previous Corollary 3.4, we conclude that $\iota\otimes 1_M$ is a strict Γ-filtered monomorphism. This proves the flatness of M. $\square$

Let A be a Γ-filtered K-algebra with Γfiltration FA. Noticing every A-module can be endowed with a Γ-filtration (section 1 Example (2)), Proposition 4.5 and Proposition 4.7 enable

us to reach the main results of this section. In the text below we write p.dim to denote the projective dimension of a module, gl.dim to denote the homological global dimension of a ring, and gl.w.dim to denote the global weak dimension of a ring, respectively; and moreover, we write w.dim for the weak dimension of a module.

4.8. Theorem Let A be a Γ-filtered K-algebra with Γ-filtration FA, and let $G(A)$ be the associated Γ-graded K-algebra of A. The following statements hold, especially when $A = R/I$ and $G(A) = R/\langle \mathbf{HT}(I)\rangle$ as in Theorem 2.1.
(i) Let M be an A-module with Γ-filtration FM. Then $\mathrm{p.dim}_A M \le \mathrm{p.dim}_{G(A)}G(M)$. In particular, if $G(M)$ has a (finite or infinite Γ-graded) free resolution, then M has a (finite or infinite) free resolution.
(ii) $\mathrm{gl.dim}A \le \mathrm{gl.dom}G(A)$.
(iii) If $G(A)$ is left hereditary, then A is left hereditary.
(iv) Let M be an A-module with Γ-filtration FM. Then $\mathrm{w.dim}_A M \le \mathrm{w.dim}_{G(A)}G(M)$.
(v) $\mathrm{gl.w.dim}A \le \mathrm{gl.w.dim}G(A)$.
(vi) If $G(A)$ is a Von Neuman regular ring then so is A.

Proof (i) and (ii) are immediate consequences of Proposition 4.4 and Proposition 4.5.
(iii) If $G(A)$ is left hereditary, then every left ideal of $G(A)$ is a projective $G(A)$-module. Let L be a left ideal of A and FL the Γ-filtration of L induced by FA. Using the inclusion map $L \hookrightarrow A$ (note that this is a strict Γ-filtered A-homomorphism), we may view $G(L)$ as a Γ-graded left ideal of $G(A)$ by Corollary 3.4. Thus, $G(L)$ is a projective $G(A)$-module, and it follows from Proposition 4.4 that L is a projective A-module. Therefore, A is left hereditary.
(iv) Note that any exact sequence

$$0 \to N \to L_n \to \cdots \to L_1 \to L_0 \to M \to 0$$

consisting of Γ-filtered free A-modules L_i and strict Γ-filtered A-homomorphisms yields an exact sequence

$$0 \to G(N) \to G(L_n) \to \cdots \to G(L_1) \to G(L_0) \to G(M) \to 0$$

consisting of Γ-graded free $G(A)$-modules $G(L_i)$ and Γ-graded $G(A)$-homomorphisms, where N has the Γ-filtration FN induced by FL_n. This assertion is an immediate consequences of Proposition 4.7.
(v) and (vi) follow from (iv). □

5. With Gröbner Bases: $G^{\mathcal{B}}(R/I) \cong R/\langle \mathbf{LM}(\mathcal{G})\rangle$ & $G^{\mathbb{N}}(R/I) \cong R/\langle \mathbf{HT}(\mathcal{G})\rangle$

By using Gröbner bases in a computational setting, the aim of this section is to decode the defining relations of the associated graded K-algebra of the K-algebra R/I in Theorem 2.1 with respect to the $\mathcal{B}$-filtration and the $\mathbb{N}$-filtration of R/I, respectively.

Let $R = K[a_1, ..., a_n]$ be a finitely generated K-algebra over a field K, where R has a K-basis $\mathcal{B}$ consisting of *monomials* of the form

$$u = a_{i_1} \cdots a_{i_s}, \quad a_{i_j} \in \{a_1, ..., a_n\},\ s \in \mathbb{N},\ s \geq 1.$$

Suppose that $\mathcal{B}$ is a *skew multiplicative K-basis* of R in the sense that

$$\text{(sm)} \qquad u,\ v \in \mathcal{B} \text{ implies } \begin{cases} u \cdot v = \lambda w \text{ for some } \lambda \in K^*,\ w \in \mathcal{B}, \\ \text{or } u \cdot v = 0. \end{cases}$$

The reason that we use the word "skew" here is that free algebras, commutative polynomial algebras, the coordinate rings of quantum affine K-spaces, and path algebras defined by finite directed graphs, all are involved as the most important practical examples supporting our text.

Let $\prec$ be a total ordering on $\mathcal{B}$. If we adopt the commonly used terminology in computational algebra, then for $f \in R$, say

$$f = \sum_{i=1}^{s} \lambda_i u_i, \quad \lambda_i \in K^*,\ u_i \in \mathcal{B},\ u_1 \prec u_2 \prec \cdots \prec u_s,$$

the *leading monomial* of f, denoted $\mathbf{LM}(f)$, is defined as $\mathbf{LM}(f) = u_s$; the *leading coefficient* of f, denoted $\mathbf{LC}(f)$, is defined as $\mathbf{LC}(f) = \lambda_s$; and the *leading term* of f, denoted $\mathbf{LT}(f)$, is defined as $\mathbf{LT}(f) = \mathbf{LC}(f)\mathbf{LM}(f) = \lambda_s u_s$. Thus, for a subset S of R, the set of leading monomials of S is defined as $\mathbf{LM}(S) = \{\mathbf{LM}(f) \mid f \in S\}$.

Under the assumption (sm) on $\mathcal{B}$, recall that a *monomial ordering* on R is a *well-ordering* $\prec$ on $\mathcal{B}$ satisfying the following conditions:

(Mo1) If $u \prec v$, then $\mathbf{LM}(uw) \prec \mathbf{LM}(vw)$ if both $uw \neq 0$ and $vw \neq 0$.
(Mo2) If $u \prec v$, then $\mathbf{LM}(su) \prec \mathbf{LM}(sv)$ if both $su \neq 0$ and $sv \neq 0$.
(Mo3) If $uw = \lambda v$, then $v \succ u$ and $v \succ w$.
Besides, if $1 \in \mathcal{B}$, then it is required that $1 \prec u$ for all $u \in \mathcal{B} - \{1\}$, and moreover, $v, u, w \neq 1$ in the axiom (Mo3).

If $\prec$ is a monomial ordering on R, then, by mimicking (e.g., [Gr]), R holds a Gröbner basis theory, that is, theoretically every ideal I of R has a (finite or infinite) Gröbner basis $\mathcal{G}$ in the sense that

$$\langle \mathbf{LM}(I) \rangle = \langle \mathbf{LM}(\mathcal{G}) \rangle.$$

$\mathcal{B}$-filtered case
Let R be as fixed above. In this part we assume that R *does not have divisors of zero*, $1 \in \mathcal{B}$ (thus, path algebras are excluded), and le $\prec$ be a monomial ordering on R. Hence R holds a Gröbner basis theory with respect to $(\mathcal{B}, \prec)$.

Note that R is $\mathcal{B}$-graded, namely, $R = \oplus_{u \in \mathcal{B}} R_u$ with $R_u = Ku$. In this case, we see that for $f \in R$, the head term $\mathbf{HT}(f)$ of f defined in section 2 is the same as the leading term $\mathbf{LT}(f)$ of

f defined above, that is,

$$\mathbf{HT}(f) = \lambda_s u_s = \mathbf{LT}(f) = \mathbf{LC}(f)\mathbf{LM}(f),$$

It turns out that if I is an ideal of R, then

$$\langle \mathbf{HT}(I)\rangle = \langle \mathbf{LM}(I)\rangle,$$

where the latter is usually called the *initial monomial ideal* of I, and consequently $R/\langle \mathbf{LM}(I)\rangle$ is called the *associated monomial algebra* of the algebra R/I.

Since R has no divisors of zero, by section 1, R is $\mathcal{B}$-filtered by the $\mathcal{B}$-grading filtration $F^{\mathcal{B}}R = \{F_u^{\mathcal{B}}R\}_{u\in\mathcal{B}}$, where

$$F_u^{\mathcal{B}}R = \oplus_{v\preceq u}R_v, \quad u \in \mathcal{B}.$$

Now, let I be an ideal of R and $A = R/I$, the quotient algebra of R defined by I. Then A has the $\mathcal{B}$-filtration $F^{\mathcal{B}}A = \{F_u^{\mathcal{B}}A\}_{u\in\mathcal{B}}$ induced by FR, that is,

$$F_u^{\mathcal{B}}A = (F_u^{\mathcal{B}}R + I)/I, \quad u \in \mathcal{B},$$

which defines the associated $\mathcal{B}$-graded K-algebra $G^{\mathcal{B}}(A) = \oplus_{u\in\mathcal{B}}G^{\mathcal{B}}(A)_u$ with $G^{\mathcal{B}}(A)_u = F_u^{\mathcal{B}}A/F^{\mathcal{B}*}_uA$ (see section 1).

5.1. Theorem With notation as fixed above, let $\mathcal{G}$ be a generating set of the ideal I. The following statements are equivalent.
(i) $\mathcal{G}$ is a Gröbner basis for I with respect to the given monomial ordering $\prec$ on $\mathcal{B}$.
(ii) $\langle \mathbf{LM}(I)\rangle = \langle \mathbf{LM}(\mathcal{G})\rangle$.
(iii) $G^{\mathcal{B}}(A) \cong R/\langle \mathbf{LM}(I)\rangle = R/\langle \mathbf{LM}(\mathcal{G})\rangle$.

Proof Note that $\langle \mathbf{LM}(\mathcal{G})\rangle \subseteq \langle \mathbf{LM}(I)\rangle$. This follows immediately from the definition of a Gröbner basis in R and Theorem 2.1. $\Box$

Remark In view of Theorem 2.1, Theorem 3.1(i) and Theorem 5.1, actually, a richer Gröbner basis theory in both commutative and noncommutative cases may be introduced by solving the isomorphic problem

$$G^{\mathcal{B}}(A) \xrightarrow[?]{\cong} R/\langle \mathbf{LM}(F)\rangle$$

for a given generating set F of the ideal I. On this aspect, a systematic clarification has been done in [Li3].

$\mathbb{N}$-graded case

In this part we *allow the case* that R has divisors of zero, for instance, R is a path algebra defined by a finite directed graph.

By the choice of the K-basis $\mathcal{B}$, R is also $\mathbb{N}$-graded by the natural $\mathbb{N}$-gradation $\{R_p\}_{p\in\mathbb{N}}$ defined by lengths of elements in $\mathcal{B}$, that is, $R = \oplus_{p\in\mathbb{N}}R_p$ with

$$R_p = K\text{-span}\left\{ u = a_{i_1}^{\alpha_1}\cdots a_{i_s}^{\alpha_s} \in \mathcal{B} \;\middle|\; \alpha_1 + \cdots + \alpha_s = p\right\},\ p \in \mathbb{N}.$$

Also recall that if $f \in R$, $f = r_p + r_{p-1} + \cdots + r_0$ with $r_i \in R_i$ and $r_p \neq 0$, then the head term of f is defined as $\mathbf{HT}(f) = r_p$, and we say that f is of degree p in R, denoted $d(f) = p$. For a subset $S \subset R$, we write

$$\mathbf{HT}(S) = \left\{ \mathbf{HT}(f) \;\middle|\; f \in S \right\}.$$

Further, let $\prec$ be a well-ordering on $\mathcal{B}$. If the ordering $\prec_{gr}$ defined for $u, v \in \mathcal{B}$ by the rule

$$u \prec_{gr} v \quad \Leftrightarrow \quad \begin{array}{l} d(u) < d(v) \\ \text{or } d(u) = d(v) \text{ and } u \prec v \end{array}$$

is a monomial ordering on $\mathcal{B}$ in the sense of the foregoing (Mo1) – (Mo3), then we call $\prec_{gr}$ a *graded monomial ordering* on $\mathcal{B}$.

5.2. Theorem (A generalization of [Li1] CH.III Theorem 3.7) Let $\prec_{gr}$ be a graded monomial ordering on $\mathcal{B}$, and let I be an ideal of R. Put $J = \langle \mathbf{HT}(I) \rangle$. The following two statements hold.
(i) $\mathbf{LM}(J) = \mathbf{LM}(I)$.
(ii) Let $\mathcal{G}$ be a generating set of I. Then $\mathcal{G}$ is a Gröbner basis of I with respect to $(\mathcal{B}, \prec_{gr})$ if and only if $\mathbf{HT}(\mathcal{G})$ is a Gröbner basis for the $\mathbb{N}$-graded ideal J of R with respect to $(\mathcal{B}, \prec_{gr})$.

Proof (i) First, note that $\prec_{gr}$ is a graded monomial ordering on $\mathcal{B}$. For $f \in R$, we have

$$(*) \qquad \mathbf{LM}(f) = \mathbf{LM}(\mathbf{HT}(f)),$$

and this turns out $\mathbf{LM}(I) = \mathbf{LM}(\mathbf{HT}(I))$. Hence $\mathbf{LM}(I) \subset \mathbf{LM}(J)$. It remains to prove the inverse inclusion. Since J is an $\mathbb{N}$-graded ideal of R, noticing the formula $(*)$ above, we need only to consider the leading monomials of homogeneous elements. Let $F \in J$ be a homogeneous element of degree p. Then $F = \sum_i G_i \mathbf{HT}(f_i) H_i$, where G_i, H_i are homogeneous elements of R and $f_i \in I$, such that $d(G_i) + d(f_i) + d(H_i) = p$ whenever $G_i \mathbf{HT}(f_i) H_i \neq 0$. Write $f_i = \mathbf{HT}(f_i) + f_i'$ such that $d(f_i') < d(f_i)$. Then

$$\sum_i G_i f_i H_i = F + \sum_i G_i f_i' H_i,$$

in which $d(\sum_i G_i f_i' H_i) < p = d(F)$. Hence $\mathbf{LM}(F) = \mathbf{LM}(\sum_i G_i f_i H_i) \in \mathbf{LM}(I)$. This shows that $\mathbf{LM}(J) \subset \mathbf{LM}(I)$, and consequently, the desired equality follows.
(ii) Note that the above formula $(*)$ yields $\langle \mathbf{LM}(\mathcal{G}) \rangle = \langle \mathbf{LM}(\mathbf{HT}(\mathcal{G})) \rangle$. By the equality in (i) we have

$$\langle \mathbf{LM}(I) \rangle = \langle \mathbf{LM}(\mathcal{G}) \rangle \text{ if and only if } \langle \mathbf{LM}(J) \rangle = \langle \mathbf{LM}(\mathbf{HT}(\mathcal{G})) \rangle.$$

Hence the equivalence follows. □

Let I be an ideal of R and $A = R/I$. Then it follows from section 1 that the $\mathbb{N}$-grading filtration $F^{\mathbb{N}}R = \{F_p^{\mathbb{N}}R\}_{p \in \mathbb{N}}$ of R with $F_p^{\mathbb{N}}R = \oplus_{i \leq p} R_i$ induces the natural $\mathbb{N}$-filtration $F^{\mathbb{N}}A = \{F_p^{\mathbb{N}}A\}_{p \in \mathbb{N}}$ of A with

$$F_p^{\mathbb{N}}A = (F_p^{\mathbb{N}}R + I)/I, \quad p \in \mathbb{N},$$

that defines the associated $\mathbb{N}$-graded K-algebra $G^{\mathbb{N}}(A)$ of A, namely,

$$G^{\mathbb{N}}(A)=\bigoplus_{p\in\mathbb{N}}G^{\mathbb{N}}(A)_p \text{ with } G^{\mathbb{N}}(A)_p=F_p^{\mathbb{N}}A/F_{p-1}^{\mathbb{N}}A.$$

5.3. Theorem (A generalization of [Li1] CH.III Theorem 3.6, [Li2] Theorem 2.1) Let $\prec_{gr}$ be a graded monomial ordering on $\mathcal{B}$ and let I be an ideal of R. Consider the algebra $A=R/I$ with the natural $\mathbb{N}$-filtration $F^{\mathbb{N}}A=\{F_p^{\mathbb{N}}A\}_{p\in\mathbb{N}}$, and let $G^{\mathbb{N}}(A)$ be the associated $\mathbb{N}$-graded algebra of A. Suppose $\mathcal{G}$ is a Gröbner basis of I with respect to $(\mathcal{B},\prec_{gr})$. Then $\langle \mathbf{HT}(I)\rangle=\langle\mathbf{HT}(\mathcal{G})\rangle$ and hence

$$G^{\mathbb{N}}(A)\cong R/\langle\mathbf{HT}(I)\rangle=R/\langle\mathbf{HT}(\mathcal{G})\rangle.$$

Proof We prove the equality $\langle\mathbf{HT}(I)\rangle=\langle\mathbf{HT}(\mathcal{G})\rangle$ by showing that $\mathcal{G}$ satisfies the condition of Proposition 2.2(i). Let $f\in I$. As $\mathcal{G}$ is a Gröbner basis for I, f has a presentation

$$f=\sum_j\lambda_ju_jg_jv_j,\quad \lambda_j\in K,\ u_j,v_j\in\mathcal{B},\ g_j\in\mathcal{G},$$

satisfying $\mathbf{LM}(u_j\mathbf{LM}(g_j)v_j)\preceq_{gr}\mathbf{LM}(f)$ for all $u_jg_jv_j\neq 0$ this is a result by division by $\mathcal{G}$). Since $\prec_{gr}$ is a graded monomial ordering on $\mathcal{B}$, for any $h\in R$ we have $\mathbf{LM}(h)=\mathbf{LM}(\mathbf{HT}(h))$ and $d(\mathbf{LM}(h))=d(\mathbf{HT}(h))=d(h)$. Thus, the presentation of f yields

$$d(u_j)+d(g_j)+d(v_j)\le d(f) \text{ for all } u_jg_jv_j\neq 0,$$

as desired. Now it follows from Theorem 2.1 that $G^{\mathbb{N}}(A)\cong R/\langle\mathbf{HT}(I)\rangle=R/\langle\mathbf{HT}(\mathcal{G})\rangle$. $\square$

By Theorem 5.2(i), in the case that a graded monomial ordering $\prec_{gr}$ is used, we see that if R/I has the $\mathcal{B}$-filtration induced by the $\mathcal{B}$-grading filtration FR of R, then

$$G^{\mathcal{B}}(R/I)\cong R/\langle\mathbf{LM}(I)\rangle\cong G^{\mathcal{B}}(R/\langle\mathbf{HT}(I)\rangle=G^{\mathcal{B}}(G^{\mathbb{N}}(R/I)).$$

Combining the classical lifting technics for $\mathbb{N}$-filtered algebras, we now summarize the whole lifting strategy of sections 2 – 4 with respect to both $\mathcal{B}$-filtration and $\mathbb{N}$-filtration in the following diagram:

$$\begin{tikzcd}
 & R/I & \\
G^{\mathcal{B}}(R/I) \arrow[ur, "\text{lifting}"] & & G^{\mathbb{N}}(R/I)=\dfrac{R}{\langle\mathbf{HT}(I)\rangle}=\dfrac{R}{\langle\mathbf{HT}(\mathcal{G})\rangle} \arrow[ul, "\text{lifitng (classical)}"'] \\
\dfrac{R}{\langle\mathbf{LM}(\mathcal{G})\rangle}=\dfrac{R}{\langle\mathbf{LM}(I)\rangle} \arrow[u, "\cong"] \arrow[rr, "\cong"'] & & G^{\mathcal{B}}\left(\dfrac{R}{\langle\mathbf{HT}(I)\rangle}\right) \arrow[u, "\text{lifting}"']
\end{tikzcd}$$

6. The first application

As the first application of Theorem 5.1 and Theorem 5.3, we now can mention a result that generalizes several well-known facts.

Let $R = K[a_1, ..., a_n]$ and the skew multiplicative K-basis $\mathcal{B}$ of R be as in section 5. Suppose that $\prec$ is a monomial ordering on $\mathcal{B}$. If I is an ideal of R, then by the fundamental decomposition theorem for the vector space R, we have

$$R = I \oplus K\text{-span}(\mathcal{B} - \mathbf{LM}(I)) = \langle \mathbf{LM}(I) \rangle \oplus K\text{-span}(\mathcal{B} - \mathbf{LM}(I)).$$

6.1. Corollary With assumption as made above, Let $A = R/I$, and let $F^{\mathcal{B}}A$ be the $\mathcal{B}$-filtration of A (if it exists) and $F^{\mathbb{N}}A$ the natural $\mathbb{N}$-filtration of A. Write $G^{\mathcal{B}}(A)$ and $G^{\mathbb{N}}(A)$ for the associated graded algebras of A with respect to $F^{\mathcal{B}}A$ and $F^{\mathbb{N}}A$, respectively. The following statements hold.
(i) The image of the set $\mathcal{B}-\mathbf{LM}(I)$ in $A = R/I$, $G^{\mathcal{B}}(A) = R/\langle \mathbf{LM}(I) \rangle$, and $G^{\mathbb{N}}(A) = R/\langle \mathbf{HT}(I) \rangle$, respectively, serves to give a K-basis for each algebra listed.
(ii) $A = R/I$ is finite dimensional over K if and only if $G^{\mathcal{B}}(A) = R/\langle \mathbf{LM}(I) \rangle$ is finite dimensional over K if and only if $G^{\mathbb{N}}(A) = R/\langle \mathbf{HT}(I) \rangle$ is finite dimensional over K, and in this case we have

$$\dim_K A = \dim_K G^{\mathcal{B}}(A) = \dim_K G^{\mathbb{N}}(A) = |\mathcal{B} - \mathbf{LM}(I)|.$$

(iii) Consider the natural $\mathbb{N}$-filtration for $A = R/I$, $G^{\mathcal{B}}(A) = R/\langle \mathbf{LM}(I) \rangle$, and $G^{\mathbb{N}}(A) = R/\langle \mathbf{HT}(I) \rangle$,respectively. Then all three K-algebras have the same Hilbert function, and hence, they have the same growth, or equivalently, they have the same Gelfand-Kirillov dimension.
(iv) If I is an $\mathbb{N}$-graded ideal of R, then the $\mathbb{N}$-graded K-algebra $A = R/I$ and the $\mathcal{B}$-$\mathbb{N}$-graded K-algebra $G^{\mathcal{B}}(A) = R/\langle \mathbf{LM}(I) \rangle$ have the same Hilbert series. Hence in (iii) above, $G^{\mathcal{B}}(A) = R/\langle \mathbf{LM}(I) \rangle$ and $G^{\mathbb{N}}(A) = R/\langle \mathbf{HT}(I) \rangle$ have the same Hilbert series. (v) If $\mathcal{G}$ is a Gröbner basis of I, then the set $\mathcal{B} - \mathbf{LM}(I)$, the Gelfand-Kirillov dimension, and the Hilbert series of the respective algebra considered in (i) – (iv) above may be obtained algorithmically.

$\square$

7. Realization via Gröbner Bases and Ufnarovski Graphs

Thanks to [[An1], [An2], [G-IL], [G-I1], [G-I2], [Nor], [Uf1], and [Uf2], in this section we indicate how to realize some of the foregoing lifting properties by virtue of Gröbner bases and the associated Ufnarovski (chain) graphs. To be concrete, we focus on a free K-algebra $R = K\langle X \rangle$ with $X = \{X_1, ..., X_n\}$ and the data $(\mathcal{B}, \prec_{gr})$, where $\mathcal{B}$ is the standard K-basis and $\prec_{gr}$ is some graded monomial ordering on $\mathcal{B}$. All notations are retained as before.

Let $\Omega = \{u_1, ..., u_s\}$ be a finite *reduced* subset of $\mathcal{B}$ in the sense that u_i and u_j are not divisible each other if $i \neq j$. If $u_i \in \Omega$, then, as before we write $d(u_i)$ for the degree (length) of u_i with

respect to the $\mathbb{N}$-gradation of R. Put

$$\ell = \max\left\{d(u_i) \;\middle|\; u_i \in \Omega\right\}.$$

Then the *Ufnarovski graph* of Ω (in the sense of [Uf1]), denoted $\Gamma(\Omega)$, is defined as a directed graph, in which the set of vertices V is given by

$$V = \left\{v_i \;\middle|\; v_i \in \mathcal{B} - \langle\Omega\rangle,\ d(v_i) = \ell - 1\right\},$$

and the set of edges E contains the edge $v_i \to v_j$ if and only if there exist X_i, $X_j \in X$ such that $v_iX_i = X_jv_j \in \mathcal{B} - \langle\Omega\rangle$. The essential link between this graph and the monomial K-algebra $R/\langle\Omega\rangle$ is that there is a bijective correspondence between the set of monomials of degree $\geq \ell - 1$ in $\mathcal{B} - \langle\Omega\rangle$ and paths in the graph.

The first effective application of Ufnarovski graph was made to determine the growth of a finitely presented K-algebra, for instance,

- ([Uf1] 1982) the growth of $R/\langle\Omega\rangle$ is exponential if and only if there are two different cycles in the graph $\Gamma(\Omega)$ with a common vertex. Otherwise, $R/\langle\Omega\rangle$ has polynomial growth of degree d, where d is the maximal possible number of different cycles in $\Gamma(\Omega)$ through which one path can pass. Hence, as a K-vector space, $\dim_K(R/\langle\Omega\rangle) < \infty$ if and only if the graph $\Gamma(\Omega)$ does not contain any cycle.

In view of section 6, if $A = R/I$ is a finitely presented K-algebra (i.e., I is a finitely generated ideal of R), and if $\mathcal{G}$ is a finite reduced Gröbner basis for the ideal I (it is known that if I has a Gröbner basis, then a reduced Gröbner basis can always be obtained by using the division algorithm), then the same statement as above can be mentioned for A and its associated $\mathbb{N}$-graded algebra $G^{\mathbb{N}}(A) = R/\langle\mathbf{HT}(\mathcal{G})\rangle$ by means of the Ufnarovski graph $\Gamma(\mathbf{LM}(\mathcal{G}))$.

Let I be an arbitrary ideal of R and $\mathbf{HT}(I)$ the set of head terms of I with respect to the natural $\mathbb{N}$-gradation of R. Below we realize other lifting properties for algebras R/I and $R/\langle\mathbf{HT}(I)\rangle$ ($\cong G^{\mathbb{N}}(R/I)$).

Noetherianity

First of all, by Theorem 3.5, the following proposition is clear.

7.1. Proposition Let J be a monomial ideal of R such that $J \subseteq \langle\mathbf{LM}(I)\rangle$. If the monomial algebra R/J is left (right) Noetherian then the algebra R/I and the algebra $R/\langle\mathbf{HT}(I)\rangle$ are left (right) Noetherian.

$\square$

7.2. Theorem With notation as fixed above, let R/J be a finitely presented monomial algebra defined by the monomial ideal $J = \langle\Omega\rangle$ with $\Omega = \{u_1, ..., u_s\} \subset \mathcal{B}$ a reduced subset. Suppose

that $J \subseteq \langle \mathbf{LM}(I) \rangle$ (for instance, $\Omega \subseteq \mathbf{LM}(I)$). If there is no edge entering (leaving) any cycle of the graph $\Gamma(\Omega)$, then the algebra $A = R/I$ and $R/\langle \mathbf{HT}(I) \rangle$ are left (right) Noetherian.

Proof By ([Uf2], [Nor]), the finitely presented monomial algebra R/J is left (right) Noetherian if and only if there is no edge entering (leaving) any cycle of the graph $\Gamma(\Omega)$. So the theorem follows from Proposition 7.1.

$\square$

As a small example let us look at the monomial algebra $S = K\langle X, Y\rangle / \langle X^2, YX\rangle$. Put $\Omega = \{X^2, YX\}$. It is easy to see that the graph $\Gamma(\Omega)$ is of the form

$$\begin{array}{ccc} X & & Y \\ \bullet & \longrightarrow & \bullet \circlearrowleft \end{array}$$

and there is no edge leaving the only cycle of $\Gamma(\Omega)$. Hence S is right Noetherian. It follows from Theorem 7.3 that any algebra $A = K\langle X, Y\rangle / I$ with X^2, $YX \in \langle \mathbf{LM}(I) \rangle$ and the algebra $K\langle X, Y\rangle / \langle \mathbf{HT}(I) \rangle$ are right Noetherian, for instance, if we set $Y \prec_{gr} X$ and take $I = \langle X^2 + aY^2 + b, YX + cX + d, h(X, Y)\rangle$, where $a, b, c, d \in K$, $h(X, Y) \in K\langle X, Y\rangle$.

Remark Recall that an algebra is called *weak Noetherian* if it satisfies the ascending chain condition for ideals. If $\Omega \subset \mathcal{B}$ is a finite reduced subset, then it was proved in [Nor] that the algebra $R/\langle \Omega \rangle$ is weak Noetherian if and only if the Ufnarofski graph $\Gamma(\Omega)$ does not contain any cycle with edges both entering and leaving it. In a similar way one may also get an analogue of Theorem 7.2 on the weak Noetherianity of R/I and $R/\langle \mathbf{HT}(I) \rangle$.

Semisimplicity, primeness and semiprimeness

Let $\mathcal{G} = \{g_1, ..., g_s\}$ be a reduced Gröbner basis of I with respect to $(\mathcal{B}, \prec_{gr})$, $\mathbf{LM}(\mathcal{G})$ the set of leading monomials of $\mathcal{G}$, and let $\ell = \max\{d(u) \mid u \in \mathbf{LM}(\mathcal{G})\}$. Recall from [Uf1] and [G-I2] that a vertex in the Ufnarovski graph $\Gamma(\mathbf{LM}(\mathcal{G}))$ is called *cyclic* if it belongs to a cyclic route of $\Gamma(\mathbf{LM}(\mathcal{G}))$. For each monomial $v = x_{i_1} x_{i_2} \cdots x_{i_s}$ with $s > \ell - 1$, there is a unique route of $\Gamma(\mathbf{LM}(\mathcal{G}))$ which is defined by

$$R(v) = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_d,$$

where $d = s - \ell$ and $v_j = x_{i_{j+1}} x_{i_{j+2}} \cdots x_{i_{j+\ell}}$, $0 \le j \le d$. A monomial $v \in \mathcal{B} - \langle \mathbf{LM}(\mathcal{G}) \rangle$, $v \ne 1$, is called *cyclic* if $d(v) \le \ell - 1$ and v is a right-hand segment of a cyclic vertex in $\Gamma(\mathbf{LM}(\mathcal{G}))$, or, if $d(v) > \ell - 1$ and the route $R(v)$ is a subroute of a cyclic route.

7.3. Theorem With convention made above, the following statements hold.
(i) If any $v \in \mathcal{B} - \langle \mathbf{LM}(\mathcal{G}) \rangle$ with $1 \le d(v) \le \ell$ is cyclic, then R/I and $R/\langle \mathbf{HT}(I) \rangle$ are semiprime; If furthermore I is not an $\mathbb{N}$-graded ideal of R and R/I is artinian (for example $\dim_K(R/I) < \infty$), then R/I is semisimple artinian.

(ii) If $\Gamma(\mathbf{LM}(\mathcal{G}))$ satisfies
(a) any $v \in \mathcal{B} - \langle \mathbf{LM}(\mathcal{G})\rangle$ with $d(v) < \ell - 1$ is a right-hand segment of a vertex of $\Gamma(\mathbf{LM}(\mathcal{G}))$, and
(b) for any two vertices u and v of $\Gamma(\mathbf{LM}(\mathcal{G}))$, there exists a route from u to v,
then R/I and $R/\langle \mathbf{HT}(I)\rangle$ are prime rings; If furthermore I is not an $\mathbb{N}$-graded ideal of R and R/I is artinian , then R/I is semisimple artinian.

Proof This follows from Theorem 3.1 and ([G-I1] Theorem 2.21, 2.27 and 2.28). □

Recall that for any $p \in \mathbb{N}$, $\mathcal{B}_p = \{w \in \mathcal{B} \mid d(w) = p\}$ is a finite set. Also note that for a monomial $w \in \mathcal{B}$, the property that $w \in \mathcal{B} - \langle \mathbf{LM}(\mathcal{G})\rangle$ can be realized by division by $\mathbf{LM}(\mathcal{G})$. So from a computational viewpoint, the effectiveness of Theorem 7.4 is done (the reader is referred to [G-I] for some examples of monomial algebras that satisfy the required conditions and for the algorithms written in pseudo-code).

Finiteness of Global dimension

Let $\Omega \subset \mathcal{B}$ be a finite reduced subset. Following [An1] and [An2], Ufnarovski constructed in [Uf2] the *graph of chains* of Ω as a directed graph $\Gamma_{\mathrm{C}}(\Omega)$, in which the set of vertices V is defined as

$$V = \{1\} \cup X \cup \{\text{all proper suffixes of } u \in \Omega\},$$

and the set of edges E contains all edges

$$1 \longrightarrow x_i \text{ for every } x_i \in X$$

and edges defined by the rule

$$u, v \in V - \{1\},\ u \longrightarrow v \text{ in } E \quad \Leftrightarrow \quad \begin{array}{l} \text{there is a } \textit{unique } w \in \Omega \text{ such that} \\ uv = \begin{cases} w, \text{ or} \\ sw,\ s \in \mathcal{B} \end{cases} \end{array}$$

For $n \geq -1$, an *n-chain* in $\Gamma_{\mathrm{C}}(\Omega)$ is a word which can be read in the graph during a path of length $n+1$, starting from 1. The set of all n-chains in $\Gamma_{\mathrm{C}}(\Omega)$ is denoted by C_n. For example, $C_{-1} = \{1\}$, $C_0 = X$, and $C_1 = \Omega$.

7.4. Theorem Suppose that $\mathcal{G}$ is a finite reduced Gröbner basis of I with respect to $(\mathcal{B}, \prec_{gr})$. The following statements hold.
(i) If the chain graph $\Gamma_{\mathrm{C}}(\mathbf{LM}(\mathcal{G}))$ has no cycles, then $\mathrm{gl.dim}(R/I) \leq \mathrm{gl.dim}(R/\langle \mathbf{HT}(I)\rangle) \leq \mathrm{gl.dim}(R/\langle \mathbf{LM}(\mathcal{G})\rangle) < \infty$.
(ii) If the chain graph $\Gamma_{\mathrm{C}}(\mathbf{LM}(\mathcal{G}))$ does not contain any d-chains, then $\mathrm{gl.dim}(R/I) \leq \mathrm{gl.dim}(R/\langle \mathbf{HT}(I)\rangle) \leq \mathrm{gl.dim}(R/\langle \mathbf{LM}(I)\rangle) \leq d$.

Proof This follows from ([Uf2] Theorem 12), ([G-I2] Remark 3.14), ([An1] Theorem 4), the foregoing Theorem 4.8(ii) and section 5. □

Consider any Gröbner basis $\mathcal{G}$ in the free K-algebra $K\langle X_1, X_2, X_3\rangle$ with $\mathbf{LM}(\mathcal{G}) = \{X_2X_1, X_3X_1, X_3X_2\}$, then, the chain graph $\Gamma_{\rm C}(\mathbf{LM}(\mathcal{G}))$ looks like

$$
\begin{tikzcd}
1 \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& X_2 \arrow[d] & X_3 \arrow[l] \arrow[dl] \\
& X_1 &
\end{tikzcd}
$$

Clearly, $\Gamma_{\rm C}(\mathbf{LM}(\mathcal{G}))$ does not contain 3-chains. Hence $\text{gl.dim}(R/\langle\mathcal{G}\rangle) \leq \text{gl.dim}(R/\langle\mathbf{HT}(\mathcal{G})\rangle) \leq \text{gl.dim}(R/\langle\mathbf{LM}(\mathcal{G})\rangle) \leq 3$. In general, it follows from Theorem 7.4 that the following result holds.

7.5. Theorem Suppose that $\mathcal{G}$ is a finite reduced Gröbner basis of I with respect to $(\mathcal{B}, \prec_{gr})$ such that

$$\mathbf{LM}(\mathcal{G}) = \left\{ X_jX_i \;\middle|\; 1 \leq i < j \leq n \right\},$$

(hence R/I has a PBW K-basis). Then $\text{gl.dim}(R/I) \leq \text{gl.dim}(R/\langle\mathbf{HT}(I)\rangle) \leq \text{gl.dim}(R/\langle\mathbf{LM}(I)\rangle) \leq n$.

□

For the K-algebras $A = R/I$ and $R/\langle\mathbf{HT}(I)\rangle$, to use Theorem 4.8(ii) and the results of section 5 effectively in examining whether they have finite global dimension, one can also check algorithmically if K has a finite projective resolution over the $\mathbb{N}$-graded monomial algebra $\overline{A} = R/\langle\mathbf{LM}(I)\rangle = R/\langle\mathbf{LM}(\mathcal{G})\rangle$ (for instance, the Anick resolution). Nowadays some well-developed computer algebra systems such as BERGMAN [CU] can produce a reduced Gröbner basis $\mathcal{G}$ for I and an Anick resolution for K.

Remark (i) In [G-IL], it was pointed out that V. Borisenko proved in 1985 that a finitely presented monomial algebra $A = R/\langle\Omega\rangle$ satisfies some polynomial identity if and only if the Ufnarovski graph $\Gamma(\Omega)$ of Ω has no multiple vertex, and an algorithm was given to recognize this criterion. We have not yet explored what will happen to an algebra if its associated monomial algebra is a PI algebra.
(ii) We have not yet explored, on the basis of section 5, what will happen to an algebra if its associated monomial algebra is a Koszul algebra. In [Li2], only very little about this topic was discussed.
(iii) We have not yet discussed any realization of the lifting properties for quotient algebras of commutative polynomial algebras and the coordinate rings of quantum affine K-spaces, either.

We leave these tasks for the successive work.

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